1^{st} Semester M-Tech Microwave & TV Engineering And Signal Processing

Tutorial 13: Convergence of Random Sequences, Ergodicity, KL expansion

1. Let X be uniformly distributed in (o,1) and for n=1,2,..., let X_n be uniformly distributed on $(0, 1 + \frac{1}{n})$. Prove that X_n converges in distribution to X as $n \to \infty$.

   Sol:

$$F_{X_n}(x_n) = \begin{cases} 0 & ; x_n < 0 \\ \frac{x_n}{1+\frac{1}{n}} & ; 0 \leq x_n \leq 1 + \frac{1}{n} \\ 1 & ; x_n > 1 + \frac{1}{n} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & ; x \leq 0 \\ x & ; 0 < x \leq 1 \\ 1 & ; x > 1 \end{cases}$$

$$F_{X_n}(x_n) = \begin{cases} 0 & ; x_n < 0 \\ x_n & ; 0 \leq x_n \leq 1 \\ 1 & ; x_n > 1 \end{cases}$$
$$= F_X(x)$$

   $\therefore X_n \xrightarrow{d} X$ as $n \to \infty$

2. Let $\{X_n : n \geq 1\}$ be a random variable uniform on $(\frac{1}{n}, 2)$ and let X be uniform on (0,2). Prove that X_n converges in distribution to X as $n \to \infty$.

Sol:

$$F_{X_n}(x_n) = \begin{cases} 0 & ; x_n < \frac{1}{n} \\ \frac{x_n - \frac{1}{n}}{2-\frac{1}{n}} & ; \frac{1}{n} \leq x_n \leq 2 \\ 1 & ; x_n > 2 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & ; x < 0 \\ \frac{x}{2} & ; 0 \leq x \leq 2 \\ 1 & ; x > 2 \end{cases}$$

$$\begin{aligned} F_{X_n}(x_n) &= \begin{cases} 0 & ; x_n < 0 \\ \frac{x_n}{2} & ; 0 \leq x_n \leq 2 \\ 1 & ; x_n > 2 \end{cases} \\ &= F_X(x) \end{aligned}$$

$\therefore X_n \underset{\rightarrow}{d} X$ as $n \rightarrow \infty$

3. Let S_n=min$(X_1, X_2, ..., X_n)$, where $\{X_k : k \geq 1\}$ is a sequence of independent random variables, each one uniform on (0,1). Prove that the sequence $\{S_n : n \geq 1\}$ converges to 0 in mean square and probability.

Sol:
$S_n = min(X_1, X_2, ...X_n)$

$$\begin{aligned} P(S_n > s) &= P(X_1 > s).P(X_2 > s)...P(X_n > s) \\ &= (1 - S)^n \end{aligned}$$

$\therefore F_{S_n}(s) = 1 - (1-S)^n$

$$\begin{aligned} f_{S_n}(s) &= -n(1-s)^{n-1}(-1) \\ &= n(1-s)^{n-1}; O \le s \le 1 \end{aligned}$$

$$\begin{aligned} E[|S_n - 0|^2] &= E[s_n^2] \\ &= \int_{-\infty}^{\infty} s^2 f_{S_n}(s) ds \\ &= \int_0^1 s^2 n(1-s)^{n-1} ds \end{aligned}$$

Put $u = 1 - s; du = -ds; s = 0 \Rightarrow u = 1 \ \& \ s = 1 \Rightarrow u = 0$

$$\begin{aligned} \therefore E[s_n^2] &= \int_0^1 (1-u)^2 n u^{n-1} du \\ &= n \int_0^1 (1 - 2u + u^2) u^{n-1} du \\ &= \int_0^1 (u^{n-1} - 2u^n + u^{n+1}) du \\ &= n \left[\frac{u^n}{n} - 2\frac{u^{n+1}}{n+1} + \frac{u^{n+2}}{n+2} \right]_0^1 \\ &= n \left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} \right] \\ &= 1 - \frac{2}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} \end{aligned}$$

$$\lim_{n\to\infty} E[|S_n^2|] = \lim_{n\to\infty} (1 - 2 + 1) = 0$$

$\therefore E[|S_n^2|] \to 0$ as n $\to \infty \Rightarrow S_n \xrightarrow{m.s} 0$ as $n \to \infty$

$P(S_n > \varepsilon) = 1 - P(S_n \le \varepsilon) = (1-\varepsilon)^n$

$\lim_{n\to\infty} P(S_n > \varepsilon) = \lim_{n\to\infty} (1-\varepsilon)^n = 0$

$\therefore S_n \underset{\rightarrow}{p} 0$ as n$\rightarrow \infty$

Also convergence in m.s $\Rightarrow$ convergence in probability

4. Let $X_1, X_2, ..., X_n, ...$ be independent r.v. uniform on (0,1) and let Z_n= max$(X_1, X_2, ..., X_n)$.

   (a) Prove that $P(Z_n \leq z)$=z^n,$0 < z < 1$.

   (b) Let U_n=n(1-Z_n). Prove that the distribution function of U_n converges to the Exponential distribution with $\lambda = 1$ as $n \rightarrow \infty$.

   Sol:
   $X_1, X_2, ..., X_n$ are iid RV$\sim U(0,1)$
   $Z_n = max(X_1, X_2, ..., X_n)$
   $a)P(Z_{n\leq z}) = P(X_1 \leq z)P(X_2 \leq z)...P(X_n \leq z) = z^n; 0 < z < 1$
   $b)U_n = n(1 - Z_n) = g(Z_n)$
   $g'(Z_n) = -n$
   $z_1 = 1 - \frac{U_n}{n}$

$$\begin{aligned} f_{U_n}(U_n) &= \frac{f_{Z_n}(z1)}{|g'(z1)|} \\ &= \frac{f_{Z_n}(1 - \frac{u_n}{n})}{n} \\ &= \frac{n}{n}\left(1 - \frac{U_n}{n}\right)^{n-1} \\ &= \left(1 - \frac{U_n}{n}\right)^{n-1} \end{aligned}$$

$$F_{U_n}(U_n) = \int_0^{U_n} \left(1 - \frac{U_n}{n}\right)^{(n-1)} dU_n$$

Put $x = 1 - U_n \frac{1}{n}; dx = \frac{-1}{n} dU_n; U_n = 0 \Rightarrow x = 1; U_n = U_n \Rightarrow x = 1 - \frac{U_n}{n}$

$$F_{U_n}(U_n) = -n \int_1^{1-\frac{U_n}{n}} x^{n-1} dx$$
$$= (-x^n)_1^{1-\frac{U_n}{n}}$$
$$= 1 - \left(1 - \frac{U_n}{n}\right)^n$$

$$\lim_{n\to\infty} F_{U_n}(U_n) = \lim_{n\to\infty} 1 - \left(1 - \frac{U_n}{n}\right)^n$$
$$= 1 - e^{-U_n}; \because \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

$\Rightarrow U_n$ converges to exponential distribution with $\lambda = 1$ as $n \to \infty$

5. Show that if$a_n \to a$ and $E|x_n - a_n| \to 0$,then$x_n \to a$ in the MS sense as $n \to \infty$
   solution:

$$\begin{aligned}
E[|x_n - a|] &= E[(x_n - a)^2] \\
&= E[(x_n - a_n)^2 + 2(x_n - a_n)(a_n - a) + (a_n + a)^2] \\
&= E[(x_n - a_n)^2] + 2(a_n - a)E[x_n - a_n] + (a_n - a)^2 \\
\lim_{n\to\infty} E[|x_n - a|^2] &= \lim_{n\to\infty} E[|x_n - a_n|^2] + 2(a_n - a)E[(x_n - a_n)] + (a_n - a)^2 \\
&= 0 + 0 + 0
\end{aligned}$$

6. Show that the process $ae^{j(\omega t+\phi)}$ is not correlation-ergodic where a and ϕ are random variables.
   solution:

$x(t) = ae^{j(\omega t+\phi)}$; a and$\phi$ are random vaiables

$$\begin{aligned}
ensembleACF, R_{xx}(t_1,t_2) &= E[ae^{j(\omega t_1+\phi)}.a^* e^{-j(\omega t_2+\phi)}] \\
&= E[|a|^2 \, e^{j\omega(t_1+t_2)}] \\
R_{xx}(\tau)] &= E[|a|^2 \, e^{j(\omega\tau)}] \\
&= e^{j(\omega\tau)}.E[|a|^2] \\
TimeaverageACF, R'_{xx}(\tau) &= \frac{1}{2T}\int_{-T}^{T} x(t+\lambda).x^*(t)dt \\
&= \frac{1}{2T}\int_{-T}^{T} ae^{j(\omega t+\lambda)+\phi}.ae^{-j(\omega t+)\phi}dt \\
&= \frac{1}{2T}|a|^2 \int_{-T}^{T} e^{j\omega\tau}dt \\
&= |a|^2 \, e^{j\omega\tau}
\end{aligned}$$

since a is a RV ,$E[|a|^2] \neq |a|^2$ and $R_{xx}(\tau) \neq R'_{xx}(\tau)$ $\therefore$ $x(t)$ is not correlation - ergodic.

7. Show that if $C(t+\tau,t) \to 0 as t \to \infty$ uniformly in t ;then x(t) is mean ergodic.
   solution:

   if$\frac{1}{2T}\int_{-2T}^{2T}(1-\frac{|\tau|}{2T})C_{xx}(\tau)d\tau = 0$ as $\tau \to \infty$,then x(t) is mean ergodic.

   if $C_{xx}(t+\tau,t) = C_{xx}(\tau) \to 0$ as $\tau \to \infty$, them both terms inside the integral vanish to (1-0) and(0) respectively.

   $\therefore$ Total integral $\to$ 0,hence x(t) is mean ergodic.

8. Determine the K-L expansion of the wiener process w(t),autocorrelation of which is given by,

$$R(t_1,t_2) = \alpha min(t_1,t_2) = \begin{cases} \alpha t_2, & t_2 < t_1 \\ \alpha t_1, & t_2 > t_1 \end{cases}$$

   solution:

$$R(t_1,t_2) = \alpha min(t_1,t_2) = \begin{cases} \alpha t_2; & t_2 < t_1 \\ \alpha t_1; & t_2 > t_1 \end{cases}$$

$$\alpha \int_0^{t_1} t_2 \phi(t_2) dt_2 + \alpha t_1 \int_{t_1}^{T} \phi(t_2) dt_2 = \lambda \phi(t_1)$$

to solve this ,we evaluate appropriate end point condition and differentiate twice.

$$\phi(0) = 0 \alpha \int_{t_1}^{T} = \lambda \phi'(t_1)$$

$$\phi'(\tau) = \lambda \phi''(t_1) + \alpha \phi(t) = 0$$

solving the last equation

$$\phi_n(t) = \sqrt{\tfrac{2}{T}} sin(\omega_n t); \omega_n = \sqrt{\tfrac{\alpha}{\lambda m}} = \tfrac{(2m+1)\pi}{2\tau}$$

thus in the interval $(0, \tau)$, the wiener process can be written as a sum of sine wave.

$$\omega(t) = \sqrt{\tfrac{2}{T}} \sum_{n=1}^{\infty} c_n sin w_n t; c_n = \sqrt{\tfrac{2}{T}} \int_0^T \omega(t) sin \omega_n t dt$$

where the coefficientc_n are uncorrelated with variance $E(c_n^2) = \lambda_n$