I^{st} Semester M.Tech Microwave & TV Engineering And Signal Processing

Tutorial 14: Convergence of Random Sequences, Ergodicity, KL expansion

1. Let X be uniformly distributed in $(0,1)$ and for $n = 1,2...$, let X_n be uniformly distributed on $(0, 1 + 1/n)$. Prove that X_n converges in distribution to X as $n \to \infty$.

2. Let $\{X_n : n \geq 1\}$ be a random variable uniform on $(1/n, 2)$ and let X be uniform on $(0,2)$. Prove that X_n converges in distribution to X as $n \to \infty$.

3. Let $S_n = min(X_1, X_2, ..., X_n)$, where $\{X_k : k \geq 1\}$ is a sequence of independent random variables, each one uniform on $(0,1)$. Prove that the sequence $\{S_n : n \geq 1\}$ converges to 0 in mean square and in probability.

4. Let $X_1, X_2, ..., X_n, ...$ be independent r.v. uniform on $(0,1)$ and let $Z_n = max(X_1, X_2, ..., X_n)$.
   (a) Prove that $P(Z_n \leq z) = z^n, 0 < z < 1$.
   (b) Let $U_n = n(1 - Z_n)$.Prove that the distribution function of U_n converges to the Exponential distribution with $\lambda = 1$ as $n \to \infty$.

5. Show that if $a_n \to a$ and $E\{|x_n - a_n|^2\} \to 0$, then $x_n \to a$ in the MS sense as $n \to \infty$.

6. Show that the process $\mathbf{a}e^{j(\omega t + \phi)}$ is not correlation-ergodic where $\mathbf{a}$ and ϕ are random variables.

7. Show that if $C(t + \tau, t) \to 0$ as $t \to \infty$ uniformly in t; then $\mathbf{x}(t)$ is mean-ergodic.

8. Determine the K-L expansion of the Wiener process $w(t)$, autocorrelation of which is given by,

$$R(t_1, t_2) = \alpha min(t_1, t_2) = \begin{cases} \alpha t_2, & t_2 < t_1 \\ \alpha t_1, & t_2 > t_1. \end{cases}$$

9. Find the mean and variance of the random variable

$$\mathbf{n}_T = \frac{1}{2T} \int_{-T}^{T} \mathbf{x}(t) dt$$

   where $\mathbf{x}(t) = 10 + \mathbf{v}(t)$ for T = 5 and for T = 100. Assume that E$\{\mathbf{v}(t)\} = 0$, $R_v(\tau) = 2\delta(\tau)$.

10. Let ζ be selected at random from the interval $S = [0, 1]$ where we assume that the probability that ζ is in a subinterval of S is equal to the length of the subinterval. For $n = 1, 2, ...$ we define the following five sequences of random variables:

$$U_n(\zeta) = \frac{\zeta}{n}$$
$$V_n(\zeta) = \zeta(1 - \frac{1}{n})$$
$$W_n(\zeta) = \zeta e^n$$
$$Y_n(\zeta) = cos2\pi n\zeta$$
$$Z_n(\zeta) = e^{-n(n\zeta-1)}$$

   (a) Which of these sequences converge surely? Identify the limiting random variable in each case.
   (b) Which of these sequences converge almost surely? Identify the limiting random variable in each case.
   (c) Do the sequences $V_n(\zeta)$ and $Z_n(\zeta)$ converge in the mean square sense?

11. An urn contains 2 black balls and 2 white balls.At time n a ball is selected at random from the urn, and the color is noted. If the number of balls of this color is greater than the number of balls of the other color, then the ball is put back in the urn; otherwise, the ball is left out. Let $X_n(\zeta)$ be the number of black balls in the urn after the nth draw. Does this sequence of random variables converge?

12. Let $X(t) = a$ for all t, where A is a zero-mean, unit-variance random variable. Find the limiting value of the time average and compare with the value of ensemble average. What do we deduce from the result?

13. A Random Process $X(t)$ is defined by

$$X(t) = ACos(2\pi f_c t)$$

where A is a Gaussian distributed random variable of zero mean and variance σ_A^2. This random process is applied to an ideal integrator, producing the output

$$Y(t) = \int_0^t X(\tau)d\tau$$

Determine whether or not $Y(t)$ is ergodic.

14. The sinusoidal wave $x(t) = ACos\omega_0 t + \phi$ is of constant amplitude A, frequency ω_0, and phase ϕ and it represents the sample function of a random process $X(t)$. The phase ϕ is random, and is uniform in $(0, 2\pi)$.

   (a) Determine the probability density function of the random variable $X(t_k)$ obtained by observing the random process $X(t)$ at time t_k.
   (b) Determine the mean and autocorrelation function of $X(t)$ using ensemble-averaging
   (c) Determine the mean and autocorrelation function of $X(t)$ using time-averaging
   (d) Establish whether or not $X(t)$ is ergodic, and if yes, in which sense.