# $1^{st}$ Semester M.Tech Microwave & TV Engineering And Signal Processing

## Tutorial 2:Probability Density and Distribution Functions

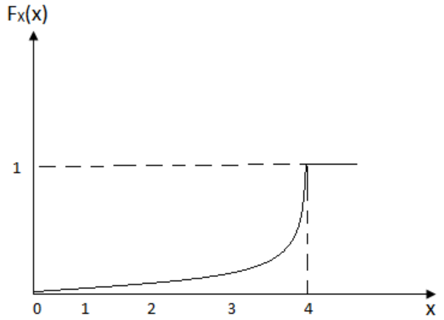
1. If the CDF of a random variable is given by $F_X(x) = \begin{cases} 0; x < 0 \\ \frac{x^2}{16}; 0 \leq x \leq 4 \\ 1; x > 4 \end{cases}$

   (a)Find $P[X > 1/X < 3]$

   (b)plot PDF

   **Solution:**

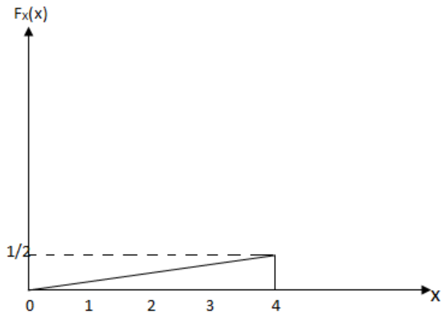
$$F_X(x) = \begin{cases} 0; x < 0 \\ \frac{x^2}{16}; 0 \leq x \leq 4 \\ 1; x > 4 \end{cases}$$




(a) $P(X > 1/X < 3)$

$$\begin{aligned} &= \frac{P((x > 1) \cap (x < 3))}{P(x < 3)} \\ &= \frac{P(1 < x < 3)}{P(x < 3)} \\ &= \frac{F_X(3) - F_X(1)}{F_X(3)} \\ &= 1 - \frac{\frac{1}{16}}{\frac{9}{16}} = \frac{8}{9} = 0.89 \end{aligned}$$

(b) $f_X(x) = \begin{cases} 0; x < 0 \\ \frac{x}{8}; 0 \leq x \leq 4 \\ 0; x > 4 \end{cases}$

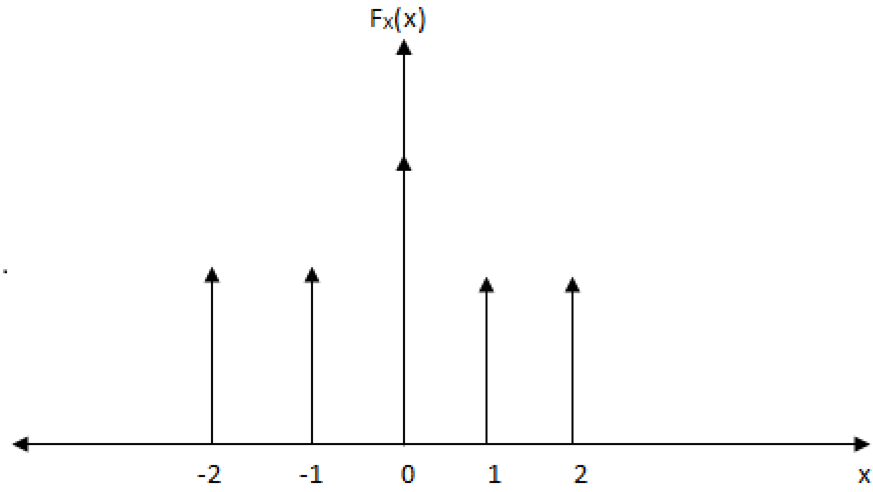


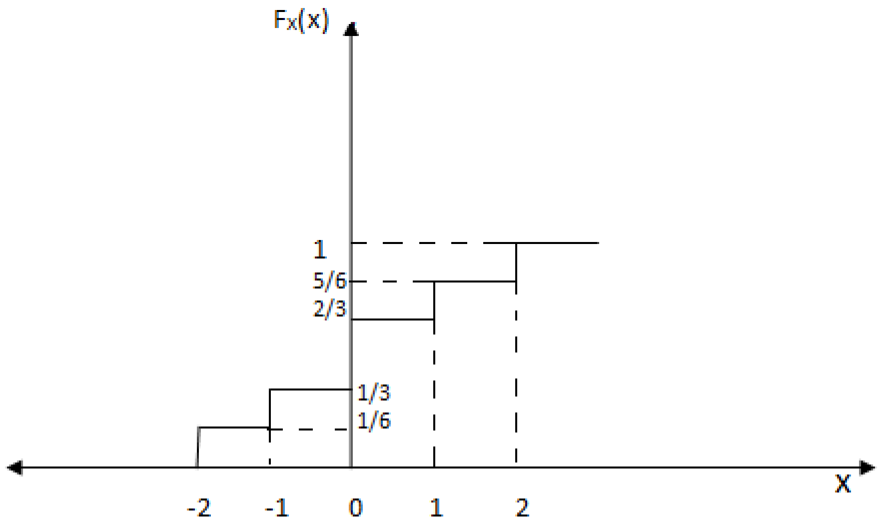

2. A random variable X takes the values-2,-1,0,1,2 such that

   $P[X = 0] = P[X > 0] = P[X < 0]$. Obtain PDF and CDF.

   **Solution:**

$$P(X = 0) = P(X < 0) = P(X > 0) = k$$
$$P(X = 0) + P(X < 0) + P(X > 0) = 1 \implies k = \frac{1}{3}$$
$$P(X > 0) = 1 - F_X(0) = \frac{1}{3} \implies F_X(0) = \frac{2}{3}$$
$$Assuming P(X = -2) = P(X = -1)$$
$$\implies P(X = -2) = P(X = -1) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$
$$Assuming P(X = 2) = P(X = 1) \implies P(X = 2) = P(X = 1) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$




3. A random variable X may assume four values with probability $\frac{1+3x}{4}, \frac{1-x}{4}. \frac{1+2x}{4}, \frac{1-4x}{4}$ Find condition on X so that these values represent the probability function of X.

**Solution:**

$\frac{1+3x+1-x+1+2x+1-4x}{4} = 1$

$\therefore \frac{4}{4} = 1$

$$\left.\begin{array}{l} 1+3x>0 \implies x>\frac{-1}{3} \\[2ex] 1+2x>0 \implies x>\frac{-1}{2} \end{array}\right\} \implies x>\frac{-1}{3}$$

$$\left.\begin{array}{l} 1-x>0 \implies x<1 \\[2ex] 1-4x>0 \implies x<\frac{1}{4} \end{array}\right\} \implies x<\frac{1}{4}$$

$\frac{-1}{3} < x < \frac{1}{4}$

4. Consider random variable X with pdf $f_X(x)$

$$f_X(x) = \begin{cases} A(1+x); -1 < x \leq 0 \\ A(1-x); 0 < x < 1 \\ 0 \qquad\quad ; \text{elsewhere} \end{cases}$$

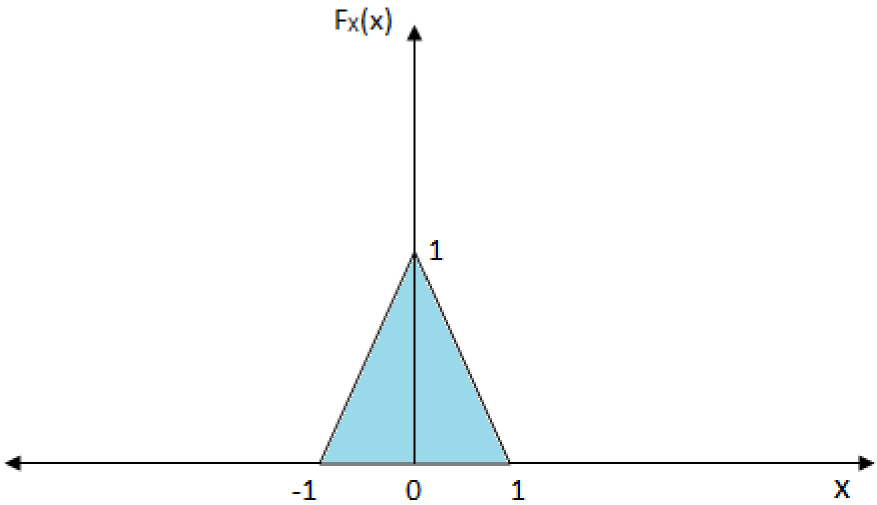
(a)Find A,plot $f_X(x)$
(b)Plot $F_X(x)$
(c)Find point b such that $P[X > b] = \frac{1}{2}P[X \leq b]$

**Solution:**

$$f_X(x) = \begin{cases} A(1+x); -1 < x \leq 0 \\ A(1-x); 0 < x < 1 \\ 0 \qquad\quad ; \text{elsewhere} \end{cases}$$

(a) $\int_{-1}^{0} A(1+x)\,dx + \int_{0}^{1} A(1-x)\,dx = 1 \implies A[x+\frac{x^2}{2}]_{-1}^{0} + [x-\frac{x^2}{2}]_{0}^{1} = 1$

$\therefore 1-\frac{1}{2}+1-\frac{1}{2} = \frac{1}{A} \implies A = 1$




(b)

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

$$-1 < x \leq 0 \implies F_X(x) = \int_{-\infty}^{x} (1+x)dx = [x + \frac{x^2}{2}]_{-1}^{x}$$

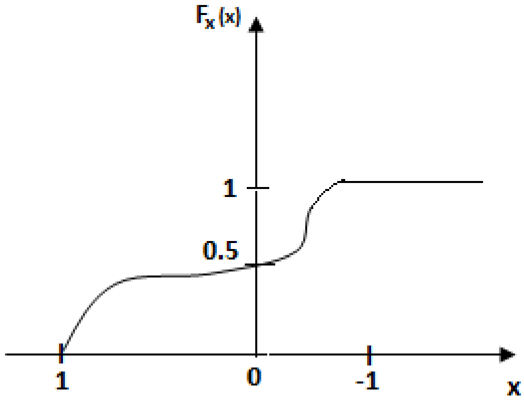
$$= x + \frac{x^2}{2} + 1 - \frac{1}{2}$$

$$= x + \frac{x^2}{2} + \frac{1}{2}$$

$$0 < x < 1 \implies F_X(x) = \frac{1}{2} + \int_{0}^{x} (1-x)dx\frac{1}{2} + [x - \frac{x^2}{2}]_{0}^{x}$$

$$= x - \frac{x^2}{2} + \frac{1}{2}$$

$$F_X(x) = \begin{cases} 0; x < -1 \\ 1; x > 1 \end{cases}$$




(c)

$$P(x > b) = \frac{1}{2}P(x \le b)$$

$$\implies 1 - F_X(b) = \frac{1}{2}F_X(b) \implies \frac{3}{2}F_X(b) = 1 \implies F_X(b) = \frac{2}{3}$$

$$\text{Since } F_X(b) > \frac{1}{2} \text{it should be part of } x > 0$$

$$x - \frac{x^2}{2} + \frac{1}{2}|_b = \frac{2}{3} \implies b^2 - 2b - 1 = \frac{2}{3}$$

$$b = \frac{6 \pm \sqrt{24}}{6} = 0.184$$

5. Let X be random variable with pdf $f_X(x) = \begin{cases} 0; x < 0 \\ ce^{-2x}; x \ge 0, c > 0 \end{cases}$

(a) Find c.
(b) Let $a > 0, x > 0$,Find $P\{X \ge (x + a)/x \ge a\}$.

**Solution:**

Comparing $f_X(x)$ with exponential distribution,

(a) c=2

(b) $P\{X \geq (x+a)/x \geq a\} = P(X \geq x) = e^{-2x}$

6. If a random variable X has pdf

$f_X(x) = \begin{cases} \frac{1}{4}; |x| < 2 \\ 0; otherwise \end{cases}$

$$\begin{aligned} Find (a) P[X < 1] \\ (b) P[|X| > 1] \\ (c) P[2X + 3 > 5] \end{aligned}$$

**Solution:**

$f_X(x) = \begin{cases} \frac{1}{4}; |x| < 2 \\ 0; otherwise \end{cases}$

$$\begin{aligned} F_X(x) &= \int_{-2}^{x} \frac{1}{4} dx; |x| < 2 \\ &= \int_{-2}^{x} \frac{1}{4} dx; x < 2 \\ &\implies F_X(x) = \frac{1}{4} x|_{-2}^{x} = \frac{1}{4}(x+2) \end{aligned}$$

(a) $P(X < 1) = F_X(1) = \frac{3}{4}$

(b)

$$\begin{aligned} P(|X|>1) &= P(\{-1<X<0\}\cap\{0>X>1\}) \\ &= P(X<-1)+P(X>1) \\ &= F_X(-1)+1-F_X(1) \\ &= \frac{1}{4}(1)+1-\{\frac{1}{4}(1+2)\} = \frac{1}{2} \end{aligned}$$

(c) $P(2X+3>5) = P(X>1) = 1-\frac{3}{4} = \frac{1}{4}$

7. In a factory 2 %of the items produced are defective, the items are packed in boxes of 100 items. What is the probability that there will be

(a) 2 defective items
(b) atleast 3 defective items
(c) $2 < defectiveitems < 5$

**Solution:**

$Since\ n = 100\ and\ p = 0.02, we\ can\ approximate\ to\ Poisson\ distribution\ with\ \lambda = np = 2$

(a) $P(X=2) = \frac{e^{-2}2^2}{2!} = 0.271$

(b) $P(x \geq 3) = P(X>2) = 1-F_X(2) = 1-\sum_{k=0}^{2} P(X=k)$

$= 1-\{\frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}+\frac{e^{-2}2^2}{2!}\}$

$= 1-e^{-2}(1+2+2)$

$= 0.323$

(c) $P(2<X<5) = P(X=3)+P(X=4)$

$= e^{-2}\{\frac{2^3}{3!}+\frac{2^4}{4!}\}$

$= 0.271$

8. Suppose 2%of the people on average are left handed. Find

(a)probability of finding 3 or more left handed

(b)probability of finding none or one left handed

**Solution:**
Since n is very large , we can approximate to Poisson distribution , with $\lambda = 0.02$

(a) $PX \geq 3 = P(X > 2) = 1 - F_X(2)$

$= 1 - \{P(X = 0) + P(X = 1) + P(X = 2)\}$

$= 1 - e^{-0.02}\{\frac{0.02^0}{0!} + \frac{0.02^1}{1!} + \frac{0.02^2}{2!}\}$

$= 1 - e^{-0.02}(1 + 0.02 + 0.0002)$

$=1.313 \times 10^{-6}$

(b) P(X=0)+P(X=1)

$= e^{-0.02}(1 + 0.02)$

=0.9998

9. The diameter of an electric cable say X is assumed to be a continous random variable with pdf $f_X(x) = 6x(1 - x) for 0 \leq x \leq 1$.Determine a number b such that $P[X < b] = P[X > b]$.Also find CDF

**Solution:**

$f_x(x) = 6 \times x \times 1 - x \qquad ; 0 \leq x \leq 1$

$F_x(x) = \int_0^x (6 \times x \times 1 - x)\ dx = 6 \times [x^2/2 - x^3/2]_0^x$

$P(x < b) = P(x > b) \implies F_x(b) = 1 - F_x(b)$
. $\qquad \implies F_x(b) = \frac{1}{2}$

$3b^2 - 2b^3 = \frac{1}{2} \implies 2b^3 - 3b^2 + \frac{1}{2} = 0 \implies 4b^3 - 6b^2 + 1 = 0$

$b = \frac{1}{2}, 1 \pm \sqrt{3}$

$\implies b = \frac{1}{2}$

10. Suppose the life hours of a radio tube as pdf $f_x(x) = \begin{cases} \frac{100}{x^2}; x \geq 5 \\ 0 \qquad ; x < 100 \end{cases}$

    (a) Find distribution function
    (b) Probability that none of three will have to be replaced during 150 hours
    (c) Probability that all the 3 tubes last for first 150 hours

    **Solution:**

    $f_x(x) = \begin{cases} \frac{100}{x^2}; x \geq 5 \\ 0 \qquad ; x < 100 \end{cases}$

    (a)$F_x(x) = \int_{100}^{x} \frac{100}{x^2}\, dx = 100 \times [\frac{x^{-1}}{-1}]_0^x$

    $=-100(\frac{1}{x} - \frac{1}{100}) = 1 - \frac{100}{x}$

    (b)$F_x(150) = 1 - \frac{100}{150} = 1 - \frac{2}{3} = \frac{1}{3} = P(X \leq 150)$

    $P(X \leq 150) = \frac{2}{3}$

    $P(\textbf{none of the tubes to be replaced}) = (\frac{2}{3})^3 = \frac{8}{27}$

    (c)$P(\textbf{all tubes last}) = P(\textbf{none to be replaced})$

    $=\frac{8}{27}$

11. We measure for resistance R of each resistor in a production line and we accept only the units the resistance of which is between 96 and 104 ohms. Find the percentage of accepted units
    (a) if R is uniform between 95 and 105 ohms.
    (b) if R is normal with $m = 100$ and $\sigma = 2ohms$.

**Solution:**

(a)$f_R(r) = \frac{1}{10}, 95 \leq x \leq 105$

$P(\textbf{R is accepted}) = \int_{96}^{104} \frac{1}{10}\,dr = \frac{8}{10} = 80\%$

(b)$f_R(r) = \frac{1}{\sqrt{2\pi\times 4}} e^{\frac{-(r-10)}{8}}$

$P(\textbf{R is accepted}) = G(\frac{104-100}{2}) - G(\frac{96-100}{2})$

$=G(2) - G(-2) = 2G(2) - 1 = 2[\frac{1}{2} + erf(2) - 1]$

$=0.954$

12. Show that if $\beta t = f(t|_{X>t})$ is the conditional failure rate of the random variable X and

$\beta(t) = kt$,then f(x) is a Rayleigh density.

**Solution:**

$F(x|_{X>t}) = \frac{P(X\leq x, X>t)}{P(X>t)}$

$\{X \leq x \cap X > t\} = \phi \qquad \text{if} \quad x < t$

$= \{t < X \leq x\} \qquad \text{if} \quad x < t$

$\therefore F(x|_{X>t}) \qquad = \frac{P(t<X\leq x)}{P(X>t)}$

$= \frac{F_X(x)-F_X(t)}{1-F_X(t)}$

$f_X(x|_{X>t}) = \frac{f_X(x)}{1-F_X(t)}$

$f_X(t|_{X>t}) = \frac{f_X(t)}{1-F_X(t)} = \beta(t) = kt$

$\Longrightarrow \frac{f_X(t)}{1-F_X(t)} = kt$

Put $1 - F_X(x) = R(x) \Longrightarrow - f_X(x) = R'(x)$

$$=\frac{-R'(x)}{R(x)} = kx$$

$$\therefore \int_0^x kx\ dx = \int_0^x \frac{-R'(x)}{R(x)}\ dx = -ln\{R(x)\}$$

$$R(x)\ = e^{-\int_0^x kx\ dx} \Longrightarrow F_X(x) = 1 - e^{-\int_0^x kx\ dx}$$

$$f_X(x) = -e^{-\int_0^x kx\ dx} \times (-kx)$$
$$=kx\ e^{-\int_0^x kx\ dx} = kx\ e^{\frac{kx^2}{2}}; x \leq 0$$

when k=$\frac{1}{\sigma^2} f_X(x)$ is Rayleigh distribution

13. If x is N(0; 2) find
(a)$P[1 \leq x \leq 2]$,
$(b)P[1 \leq x \leq 2|x \geq 1]$

**Solution:**

$X \longrightarrow N(0,2)$

(a) $P(1 \leq X \leq 2) = G(\frac{2}{\sqrt{2}}) - G(\frac{1}{\sqrt{2}})$

$$=G(\frac{2}{\sqrt{2}}) - G(\frac{1}{\sqrt{2}}) = \frac{1}{2} + erf(1.4) - \frac{1}{2} - erf(2)$$

=0.161

(b) $P(1 \leq X \leq 2|_X \geq 1) = \frac{G(\frac{2}{\sqrt{2}})-G(\frac{1}{\sqrt{2}})}{1-G(\frac{1}{\sqrt{2}})}$

=0.67

14. 20%of items produced from a factory are defective.Find the probability that in a sample of 5 chosen at a random
(a) none is defective
(b) one is defective
(c)$1 <$ defective items $< 4$

**Solution:**

p=0.2 ; q=0.8 ; n=5

(a)P(X=0)=$\begin{pmatrix} 5 \\ 0 \end{pmatrix}(0.2)^0(0.8)^5 = 0.328$

(b)P(X=1)=$\begin{pmatrix} 5 \\ 1 \end{pmatrix}(0.2)^1(0.8)^4 = 0.4096$

(c)$P(1 < X < 4) = P(X=2) + P(X=3) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}(0.2)^2(0.8)^3 + \begin{pmatrix} 5 \\ 3 \end{pmatrix}(0.2)^3(0.8)^2$

$= 0.256$

15. A continuous random variable X has the pdf
$f_X(x) = \frac{k}{1+x^2}\ ;\ -\infty < x < \infty$

find
(a) k
(b)distribution function of X
(c)$P[X \geq 0]$

**Solution:**

$f_X(x) = \frac{k}{1+x^2}\ ;\ -\infty < x < \infty$

(a)$\int_{-\infty}^{\infty} \frac{k}{1+x^2}\ dx = 1 \Longrightarrow [k tan^{-1}x]_{-\infty}^{\infty} = 1$

$\Longrightarrow\ \frac{\pi}{2} + \frac{\pi}{2} = \frac{1}{k}\ \Longrightarrow\ k = \frac{1}{\pi}$

(b)$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+x^2)}\ dx = [\frac{1}{\pi} tan^{-1}x]_{-\infty}^{x} = \frac{1}{\pi}(tan^{-1}x + \frac{\pi}{2})$

(c)$P(X \geq 0) = 1 - F_X(0) = 1 - \frac{1}{2} = \frac{1}{2}$