# $I^{st}$ Semester M.Tech Microwave & TV Engineering And Signal Processing

Tutorial 3 : A Function of a Single Random Variable

1. Resistance R in the circuit shown in $Fig1.0$ is random and has a triangular distribution, as shown in $Fig1.1$.With a constant current $i = 0.1A$ and a constant resistance $r_0 = 100\Omega$;determine the pdf of voltage V

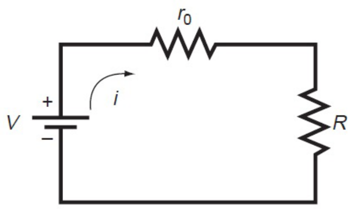



Fig. 1.0.

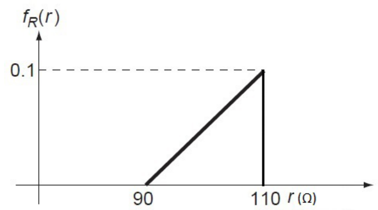


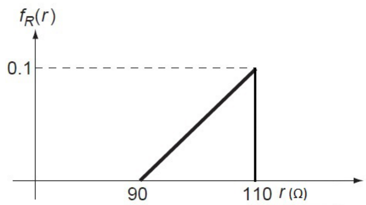

Fig. 1.1

Soln:

V=i(R+100)

=0.1(R+100)

=0.1R+10=g(R)




Root,r=$\frac{v-10}{0.1}$=10v-100

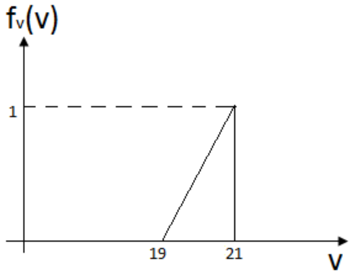
$r_1 = 10v - 100$

$g^{'}(r) = 0.1$=$g^{'}(r1)$

$f_R(r)$=0.1+$\frac{x}{200} - \frac{11}{20}$

=0.005r-0.45

$f_V(v)$=$\frac{f_R(r1)}{|g^{'}(r_1)|}$ =$\frac{0.005(10v-100)-0.45}{0.1}$



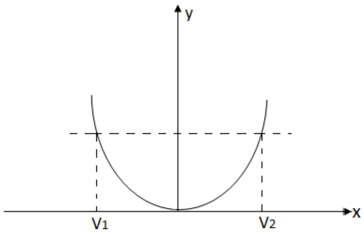

$= 0.5v - 9.5; 19 < v < 21$

2. According to Maxwell-Boltzmann law of theoretical Physics, the pdf of V, velocity of a gas molecule is given by:

$$f_V(v) = \begin{cases} kv^2e^{-av^2}; v > 0 \\ 0 \qquad\qquad ; otherwise \end{cases}$$

where $a$ is a constant depending on its mass and the absolute temperature and $k$ is an appropriate constant.Show that kinetic energy Y $= \frac{1}{2}mV^2$ is a random variable having *Gamma Distribution*
Soln:

$$f_V(v) = \begin{cases} kv^2e^{-av^2}; v > 0 \\ 0 \qquad\qquad ; otherwise \end{cases}$$




a,k are constants

Y$=\frac{1}{2}mV^2 = g(V)$

roots are $V_1 = \sqrt{\frac{2y}{m}}, V_2 = -\sqrt{\frac{2y}{m}}$ But $f_V(v) = 0; V < 0 \Rightarrow V_2$ is not valid

$g'(V)$=$\frac{m}{2}$x2V=mV $\Rightarrow g'(V_1)$= m$\sqrt{\frac{2y}{m}}$= $\sqrt{2ym}$

$f_Y(y)$= $\frac{f_V(V_1)}{|g'(V_1)|}$ = $\frac{k\frac{2y}{m}e^{-a\frac{2y}{m}}}{\sqrt{2my}}$;y $\geq 0$

This is of the form of gamma distribution with

$$\alpha = \frac{3}{2}, \beta = \frac{m}{2a}, \frac{k}{3\sqrt{\pi}a^{\frac{3}{2}}} \frac{y^{\frac{1}{2}}}{\Gamma(\frac{3}{2})\frac{m}{2a}(\frac{3}{2}}).e^{-y\frac{m}{2a}}; y \geq 0$$

3. Let X be a *Gaussian* random variable with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}; -\infty < x < \infty$$

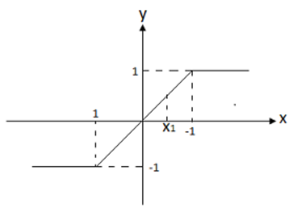
Let Y=g(X) where g(.) is the nonlinear function given as

$$g(x) = \begin{cases} -1; x < -1 \\ x; -1 \leq x \leq 1 \\ 1; x > 1 \end{cases}$$

It is called a *saturable limiter* function.(a)Sketch g(x).(b) Find$F_Y(y)$.(c)Find and sketch $f_Y(y)$.

Soln:

(a)



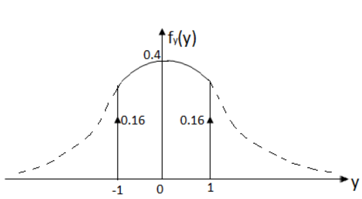

(b)$f_X(x)$=$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$; $-\infty < x < \infty$

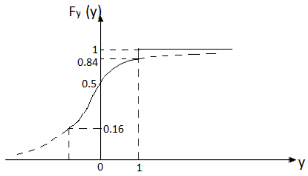
P(Y=-1) = P(X¡-1) = $F_X(-1)$ =$\frac{1}{2} - erf(1) = 0.16 = F_Y(-1)$

$P(Y = 1) = P(X > 1) = 1 - P(X \leq 1)$ = $\frac{1}{2}$-erf(1)=0.16=$F_Y(1)$

$F_Y(y)$=$F_X(y)$; $-1 < y < 1 y = x$

(c)$f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}} + 0.16\delta(y+1) + 0.16\delta(y-1); -1 < y < 1$




4. In homomorphic image processing, images are enhanced by applying non-linear transformations to the image functions.Assume that the image function is modelled as random variable X and the enhanced image Y is Y=$ln(X)$.Note that X cannot assume negative values.Compute the pdf of Y if X has an exponential density $f_X(x) = \frac{1}{3}e^{\frac{-1}{3}x}u(x)$.

   Soln:

   Y=$ln(X)$; $\quad X \geq 0$

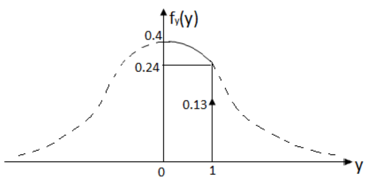
   $f_X(x) = \frac{1}{3}e^{\frac{-1}{3}x}u(x)$

   $y = ln(X) = g(x) \Rightarrow x_1 = e^y(oneroot)$

   $g^x = \frac{1}{x} \Rightarrow |g^{'}(x_1)| = \frac{1}{e^y} = e^{-y}$

   $f_Y(y) = \frac{f_X(e^y)}{e^{-y}} = e^y.\frac{1}{3}.e^{\frac{-1}{3}(e^y}; \quad -\infty < y < \infty$

5. (a)Let X:N(0,1) and let $Y_1 = g_1(X)$ where the function $g_1(.)$ is shown in $Fig$5.0.Compute $F_{Y1}(y)$ and $f_{Y1}(y) \quad from \quad f_X(x)$.




   (b)Let Z:N(0,25) be a random variable passed through the quantizer function shown in $Fig$.5.1.The output is $Y_2 = g_2(Z)$.Express the pmf of $Y_2$ in terms of the Q funtion.

   Soln:

   (a)X:N(0,1)

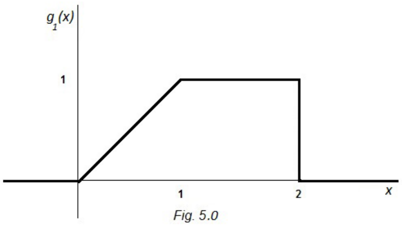


Fig. 5.0

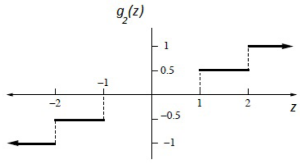
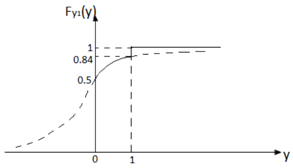


Fig. 5.1.

$$y_1 = \begin{cases} 0; x < 0 \\ x; 0 < x < 1 \\ 1; 1 < x < 2 \\ 0; x > 2 \end{cases}$$




P(Y=0) $= P(X < 0) + P(X > 2)$

$= P(X < 0) + 1 - P(X < 2)$

$= \frac{1}{2} + 1 - G(2) = 1 - erf(2) = 0.52$

$P(Y = 1) = P(1 < X \leq 2) = F_X(2) - F_X(1)$

$= erf(2) - erf(1) = 0.130 < x < 1 \Rightarrow F_{Y1}(y) = G(y)$

$f_{Y1}(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} + 0.52\delta(y) + 0.13\ \delta(y - 1)$

(b)

$$y_2 = \begin{cases} 0; -1 < z < 1 \\ -0.5; -2 < z < -1 \\ -1; z < -2 \\ 0.5; 1 < z < 2 \\ 1; z > 2 \end{cases}$$

P($Y_2$=0)=P($-1 < Z < 1$ )=1-Q($\frac{1}{5}$)-1-Q($\frac{-1}{5}$)

=1-Q($\frac{1}{5}$)-Q($\frac{1}{5}$)=1-2Q($\frac{1}{5}$)

P($Y_2$=-0.5)=$P(-2 < Z < -1)$=1-Q($\frac{-1}{5}$)-1-Q($\frac{-2}{5}$)

Q($\frac{1}{5}$)-Q($\frac{2}{5}$)

P($Y_2$=-1)=$P(Z < -2)$ =1-Q($\frac{-2}{5}$)=Q($\frac{2}{5}$)

P($Y_2$=0.5)=$P(1 < Z < 2)$=1-Q($\frac{2}{5}$)-1-Q($\frac{1}{5}$)

=Q($\frac{1}{5}$)-Q($\frac{2}{5}$)

P($Y_2$=1)=$P(Z > 2) = 1 - P(Z < 2)$ =1-1-Q($\frac{2}{5}$)

=Q($\frac{2}{5}$)

6. Let X be a uniform random variable on [0,2].Compute the pdf of Y if Y =g(X) and g(.) is as shown in *Fig*.6

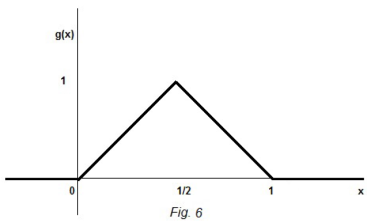


*Fig. 6*

Soln:
X:u(0,2)

y-1=$\frac{x-0.5}{-0.5}$=-2(x-0.5)

y=-2x+2; $0.5 < x < 1$

=2x;  $0 < x < 0.5$

$0 < x < 0.5 \Rightarrow f_Y(y) = \frac{f_X(y/2)}{2} = \frac{1}{4}; 0 < y < 1$

$0.5 < x < 1 \Rightarrow g(x) = -2x + 2; g^{'}(x)$=-2

$x_1$=$\frac{-1}{2}$(y-2) $\Rightarrow g^{'}(x_1)$=-2

$f_Y(y)$=$\frac{f_X \frac{-1}{2(y-2)}}{2}$=$\frac{1}{4}$;$0 < y < 1$

$P(Y = 0) = P(X < 0) + P(X > 1) = P(X < 0) + 1 - P(X < 1)$

=0+1-$\frac{1}{2}$=$\frac{1}{2}$

$f_Y$(y)=$\frac{1}{4}$+$\frac{1}{4}$+$\frac{1}{2}\delta(y)$=$\frac{1}{2}$+$\frac{1}{2}\delta(y)$;$0 < y < 1$;

$F_Y$(y)=$\frac{1}{2}$y+$\frac{1}{2}$u(y)

7. Show that if $Y = X^2$,then

$f_Y(y/X \geq 0)$=$\frac{U(y)}{1-F_X(0)}\frac{f_X(\sqrt{y}}{2\sqrt{y}}$.
Soln:
Y=$X^2$=g(X)⇒ $g^{'}(X)$=2X

$x_1$=$\sqrt{y}$,$x_2$=-$\sqrt{y}$

$F_Y(y/x \geq 0)$=$\frac{P(Y\leq y, X\geq 0)}{P(X\geq 0)}$ = $\frac{P(X^2\leq y, X\geq 0)}{P(X\geq 0}$)

when $y < 0 \quad X^2 \leq y \cap X \geq 0$=$\phi$

y¿0, $X^2 \leq y \bigcap$ X≥ 0= 0≤ X ≤ $\sqrt{y}$

$F_Y(y/x \geq 0)$ = $\frac{P(0\leq X\leq\sqrt{y})}{P(X\geq 0)}$ = $\frac{F_X(\sqrt{y})-F_X(0)}{1-F_X(0)}$;y≥ 0
$f_Y(y/x \geq 0)$=$\frac{1}{1-F_X(0)}$. $\frac{f_X(\sqrt{y})}{2\sqrt{y}}$.u(y)

8. (a)If the random variable X is uniformly distributed in $(-\pi, \pi)$, find the pdf of Y=$aSin(X+\alpha)$;where $a > 0 \quad and \quad \alpha$ are constants.
(b)The horizontal range of a periodic is given by R=$\frac{v^2}{g}Sin(2\theta)$.If $\theta$ is uniformly distributed in (0,$\frac{\pi}{2}$) and $\frac{v^2}{g}$ is constant, find the pdf of R.
Soln:
(a)X:u(-$\pi, \pi$)

Y=aSin(X+$\alpha$)=g(X);a¿0

a and $\alpha$ are constants

Two roots are:

$X_1$+$\alpha$ =$Sin^{-1}$($\frac{y}{a}$ ⇒ $X_1$=$Sin^{-1}$($\frac{y}{a}$-$\alpha$)

$X_2$+$\alpha$=$\pi$ -$Sin^{-1}$($\frac{y}{a}$ ⇒ $X_2$=$\pi$-$Sin^{-1}$($\frac{y}{a} - \alpha$)

$g^{'}(X)$=aCos(X+$\alpha$)

$g^{'}(X_1)$=$aCosSin^{-1}$($\frac{y}{a}$)=a$\sqrt{1-\frac{y^2}{a^2}}$=$\sqrt{a^2-y^2}$

—$g^{'}(X_2)$|=—aCos($\pi$ -$Sin^{-1}$($\frac{y}{a}$)—=$|-aCosX_1|$=$\sqrt{a^2-y^2}$

$f_Y(y)$=$\frac{f_X(Sin^{-1}(\frac{y}{a}-\alpha)}{\sqrt{a^2-y^2}}$

$\frac{1}{2\pi}x2x\frac{1}{\sqrt{a^2-y^2}}$ = $\frac{1}{\pi\sqrt{a^2-y^2}}$;$-a < y < a$

0 elsewhere

(b)R=$\frac{V^2}{g}Sin2\theta$= g($\theta$)

$\theta$:u(0,$\frac{\pi}{2}$) ; $\frac{V^2}{g}$=constant

r=$\frac{V^2}{g}Sin2\theta \Rightarrow \theta_1$=$\frac{1}{2}Sin^{-1}(\frac{rg}{V^2})$; $\theta_2 = \pi - \frac{1}{2}Sin^{-1}(\frac{rg}{V^2})$

$g^{'}(\theta)$=$\frac{V^2}{g}Cos2\theta(2) \Rightarrow g^{'}(\theta_1)$=$\frac{2V^2}{g}CosSin^{-1}(\frac{rg}{V^2})$=$2\sqrt{\frac{V^4}{g^2}-r^2}$

$|g^{'}(\theta_2) = 2\sqrt{\frac{V^4}{g^2}-r^2}$

$f_R(r)$=$\frac{2g}{\pi\sqrt{V^4-r^2g^2}}$ ;$0 < r < \frac{V^2}{g}$

0 elsewhere

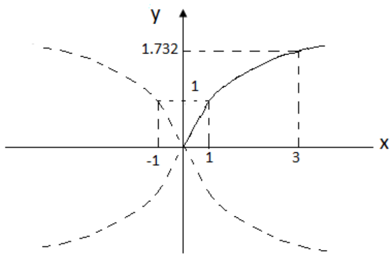
9. Let random variable X be U(-1,3).Let g(X) be Y=$\sqrt{|X|}$.Find and plot pdf of Y.
Soln:

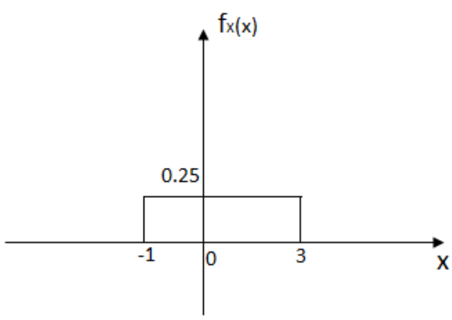
X:u(-1,3)

g(X)=Y=$\sqrt{|X|} \Rightarrow$

$$y = \begin{cases} \sqrt{x}; x > 0 \\ \sqrt{-x}; x < 0 \end{cases}$$

$$g^{'}(X) = \begin{matrix} \frac{1}{2\sqrt{x}}; x > 0 \\ \frac{-1}{2\sqrt{x}}; x < 0 \end{matrix}$$




When $0 < y < 1$, the roots are: $X_1$=$y^2$,$X_2$=$-y^2$

$g'(X_1)=\frac{1}{2\sqrt{-X_1}}=\frac{1}{2y}$

$|g'(X_2)|=|\frac{-1}{2\sqrt{-X_R}}|=|\frac{-1}{2y}|=\frac{1}{2y}$

$f_Y(y)=\frac{f_X(x_1)+f_X(x_2)}{\frac{1}{2y}}$ = y

when $1 < y < 3$, 1 root

$f_Y(y)=\frac{f_X(x_1}{\frac{1}{2y}}$=y/2

10. Suppose we are given a random variable X with range R and the CDF shown in*Fig.*10.0.What transformation Y=g(X) will make the CDF of Y as shown in *Fig.*10.1?

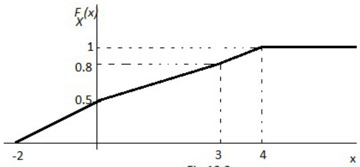



Fig 10.0

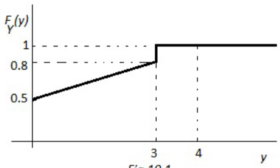
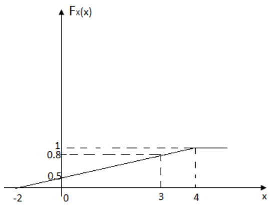
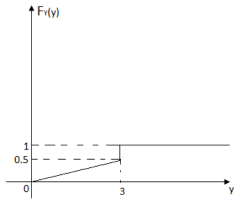
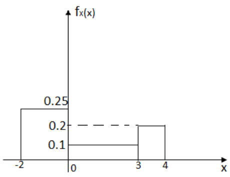


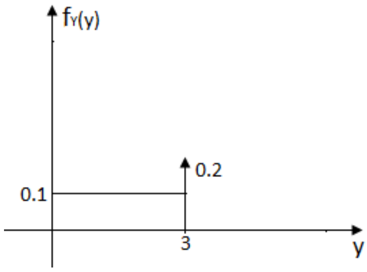

Fig 10.1

Soln:




P(Y=0)=0.5=P($X < 0$)

P(Y=3)=0.2=$F_X(4) - F_X(3)$=P($3 < X < 4$)

$\Rightarrow x < 0$ maps to y=0

$\Rightarrow 3 < x < 4$ maps to y=3

11. If X is a continuous random variable with some distribution defined over (0,1) such that $P(X \leq 0.29) = 0.75$,determine k so that $P(Y \leq K) = 0.25$,where $Y = 1 - X$.
Soln:
P(X$\leq$ 0.29)=0.75

P(Y$\leq k$)=0.25$\Rightarrow P(1 - X \leq k)$=0.25

P(-X$\leq k - 1$)=0.25$\Rightarrow P(X \geq 1 - k)$=0.25

P(X$\leq 1 - k$)=0.75$\Rightarrow 1 - k$=0.29

$\Rightarrow k$=0.71