# $1^{st}$ Semester Microwave & TV Engineering And Signal Processing

**Tutorial 4 : Mean and Variance of Random Variable**

1. A biased coin is tossed and the first outcome is is noted. The tossing is continued until the outcome is the complement of the first outcome,thus completing the first run. Let X denote the length of the first run.Find the p.m.f of X , and show that E[X] = $\frac{p}{q} + \frac{q}{p}$.

   ANS:

$$
\begin{aligned}
P(H) &= p \\
P(T) &= q \\
X : RV &= length \quad of \quad first \quad run. \quad X \quad varies \quad from \quad 1 \quad to \quad \infty \\
P(X = 1) &= P(HT) + P(TH) = pq + qp \\
P(X = 2) &= P(HHT) + P(TTH) = qp^2 + pq^2. \\
P(X = 3) &= P(HHHT) + P(TTTH) = p^3q + q^3p. \\
P(X = k) &= pq(p^{k-1} + q^{k-1}) \\
E[X] &= \sum_k P(X = k) \\
&= pq\sum_{k=1}^{\infty}(kp^{k-1} + kq^{k-1}) \\
&= \mu_1 + \mu_2 \\
\mu_1 &= pq\sum_{k=1}^{\infty} kp^{k-1} \\
&= pq + 2p^2q + 3p^3q + ... \\
&= pq(1 + 2p + 3p^2 + ...) \\
&= pq[\frac{1}{1-p} + \frac{p}{1-p} + \frac{p^2}{1-p} + ...] \\
&= p.\frac{1}{1-p} = \frac{p}{q}
\end{aligned}
$$

2. A particular color TV model is manufactured in three different plants, say, A, B and C of the same company. Because the workers at A, B, and C are not

equally experienced, the quality of the sets differ from plant to plant. The pdf's of the time-to-failure, X, in years are

$$f_X(x) = \frac{1}{5}e^{-\frac{x}{5}}u(x) for A$$
$$f_X(x) = \frac{1}{6.5}e^{-\frac{x}{6.5}}u(x) for B$$
$$f_X(x) = \frac{1}{10}e^{-\frac{x}{10}}u(x) for C$$

Plant A produces three times as many sets as B , which produces twice as many as C.The sets are all sent to a central warehouse , intermingled , and shipped to retail stores all around the country. What is the expected life time of a set purchased at random?

ANS:

$$f_{\frac{X}{A}}(x) = \frac{1}{5}e^{-\frac{x}{5}}u(x) \quad \Rightarrow \lambda_A = \frac{1}{5}$$

$$f_{\frac{X}{B}}(x) = \frac{1}{6.5}e^{-\frac{x}{6.5}}u(x) \Rightarrow \lambda_A = \frac{1}{6.5}$$

$$f_{\frac{X}{C}}(x) = \frac{1}{10}e^{-\frac{x}{10}}u(x) \quad \Rightarrow \lambda_A = \frac{1}{10}$$

$$P(A) = \frac{6}{9}; P(B) = \frac{2}{9}; P(C) = \frac{1}{9}$$

$$E[X] = E[\frac{X}{A}]P(A) + E[\frac{X}{B}]P(B) + E[\frac{X}{C}]P(C)$$

$$= \frac{1}{\lambda_A}P(A) + \frac{1}{\lambda_B}P(B) + \frac{1}{\lambda_C}P(C)$$

$$= 5 \times \frac{6}{9} + 6.5 \times \frac{2}{9} + 10 \times \frac{1}{9}$$

$$= 5.89$$

3. The negative binomial distribution with parameters N, Q, P where Q - P = 1; P > 0 and N $\geq$ 1 is defined by

$$P[X = k] = \left( \begin{array}{c} N+k-1 \\ N-1 \end{array} \right) (\tfrac{P}{Q})^k (1\text{-}\tfrac{P}{Q})^N \quad , (k = 0,1,2,...)$$

Show that the moment-generating function is $\theta_X(t)$ = (Q - P$e^t$)$^{-N}$.

ANS:

$$\phi_X(e^t) = \sum_k e^{tk} P(X = k)$$

$$P(X = k) = \left( \begin{array}{c} N+k-1 \\ N-1 \end{array} \right) (\frac{P}{Q})^k (1 - \frac{P}{Q})^N$$

$$\phi_X(e^t) = \sum_{k=0}^{\infty} e^{tk} \left( \begin{array}{c} N+k-1 \\ N-1 \end{array} \right) (\frac{P}{Q})^k (\frac{Q-P}{Q})^N$$

$$Apply, (1 - x^{-n}) = \left( \begin{array}{c} N+k-1 \\ k \end{array} \right) (-x)^k$$

$$\phi_X(e^t) = \sum_{k=0}^{\infty} (e^t \frac{P}{Q})^k \left( \begin{array}{c} N+k-1 \\ k \end{array} \right) (\frac{1}{Q})^N$$

$$= Q^{-N} (1 - e^t \frac{P}{Q})^{-N}$$

$$= (Q - Pe^t)^{-N}$$

4. A fair die is rolled 10 times. Calculate the expected sum of the 10 rolls.

ANS:

$$X \Rightarrow Sum \quad no \quad of \quad rolls.$$

$$Y_1 \Rightarrow No \quad of \quad 1^s \quad in \quad 10 \quad rolls$$

$$Y_2 \Rightarrow No \quad of \quad 2^s \quad in \quad 10 \quad rolls$$

$$\vdots$$

$$Y_6 \Rightarrow No \quad of \quad 6^s \quad in \quad 10 \quad rolls$$

$$Then X = Y_1 + 2Y_2 + ..... + 6Y_6$$

$$= \sum_{i=1}^{6} iY_i$$

$$E[X] = \sum_{i=1}^{6} iE[Y_i]$$

$$E[Y_1] = \sum_{k} Y_1 P(Y_1 = k)$$

$$= \sum_{k=0}^{10} k \begin{pmatrix} 10 \\ k \end{pmatrix} (\frac{1}{6})^k (\frac{5}{6})^{10-k}$$

$$= np = 10 \times \frac{1}{6}$$

$$= \frac{5}{3}$$

$$E[Y_i] = \frac{5}{3} \qquad for \quad every \quad i \quad since \quad all \quad have \quad equal \quad probabilities$$

$$So, E[X] = \frac{5}{3}\sum_{i=1}^{6} i = \frac{5}{3} \times 6 \times \frac{7}{2}$$
$$= 35$$

5. Suppose that A and B each randomly, and independently, choose 3 of 10 objects. Find the expected number of objects

(a)Chosen by both A and B.

ANS:

Let X be the no of objects that are selected by both A and B.Take an indicator variable, $X_i$ .

Let $X_i = 1$if object i is selected by both A & B and,
$X_i = 0$ otherwise; $1 \leq i \leq 10$

$$E[X] = E[\sum_{i=1}^{10} X_i] = \sum_{i=1}^{10} E[X_i]$$
$$E[X_i] = 1 \times P[X_i = 1] + P[X_i = 0]$$
$$= P[X_i = 1]$$
$$P[X_i = 1] = \frac{3}{10}^{2}$$
$$E[X] = \sum_{i=1}^{10} E[X_i] = 10 \times \frac{9}{100}$$
$$= 0.9$$

(b)Not chosen by both A or B.

ANS:

Let $X_i = 1$ if object i is not chosen by A and is not chosen by B.

$$P[X_i = 1] = (\frac{7}{10})^2$$
$$E[X] = \sum_{i=1}^{10} E[X_i] = \sum_{i=1}^{10} (\frac{7}{10})^2$$
$$= 10 \times (\frac{7}{100})^2 = 4.9$$

(c)Chosen by exactly one A or B.

ANS:

Here, either person A draws object i and person B doesnot or, person B doesnot or, person B draws object i and person A doeesnot.

Let $X_i$ if exactly one of A and B draws object $X_i = 0$ otherwise

$$E[X_i] = P[X_i = 1] = 2 \times \frac{3}{10} \times \frac{7}{10}$$
$$E[X] = 10 \times 2 \times \frac{3}{10} \times \frac{7}{10} = 4.2$$

6. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns 1,2,...........i. Find

(a)The expected number of urns that are empty.

ANS:

Let X be the no of empty urns. Define an indicator variable $X_i = 1$.If urn i is empty, $X_i = 0$ otherwise we must find

$$E[X_i] = P[X_i = 1]$$

$$= i(1 - \frac{1}{i})(1 - \frac{1}{1+i})(1 - \frac{1}{i+2})...(1 - \frac{1}{n}) \tag{1}$$

$$So\ E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{i-1}{n}$$

$$= \frac{1}{n} \sum_{i=1}^{n} i - 1 = \frac{1}{n} \sum_{j=0}^{n-1} j$$

$$= \frac{1}{n} \frac{n(n-1)}{2}$$

$$= \frac{n-1}{2}$$

(b)The probabilty that none of the urns is empty.

ANS:

P(urns will be empty)

$$= \frac{1}{n} . \frac{1}{n-1} . \frac{1}{n-2} ......1 = \frac{1}{n!}$$

7. Two chips are drawn at random without replacement from a box that contains five chips numbered 1 through 5.If the sum of chips drawn is even, the random variable X = 5; if the sum of chips drawn is odd, X = -3.

   (a) Find the moment-generating function (mgf) for X
   (b) Find the expected value and variance of X

8. Let X have pdf

$$f_X(x) = \begin{cases} \frac{1}{2}x^2e^{-x} & ; x \geq 0 \\ 0 & ; otherwise \end{cases}$$

   (a) Find the characteristic function of X.

   ANS:

$$\phi_x(\omega) \rightleftharpoons f_x(x).$$

$$If \quad f_{x2}(x) \quad is \quad rect_T(t),$$

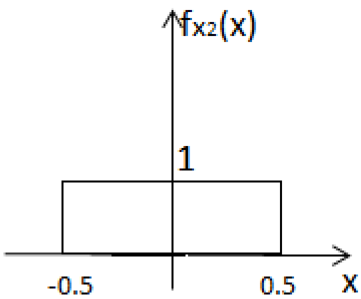
$$\phi_{x2}(\omega) = sinc(\frac{\omega}{2\pi}).\ Shift \quad by \quad \frac{1}{2} \quad to \quad get,$$

$$f_{x1}(x), \phi_{x1}(\omega) = e^{\frac{j\omega}{2}} sinc(\frac{\omega}{2\pi})$$

$$\Rightarrow \phi_{x1}(\omega) = \frac{sin\frac{\omega}{2}}{\frac{\omega}{2}} e^{\frac{j\omega}{2}} \frac{(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}})}{2j}$$

$$= \frac{e^{j\omega} - 1}{j\omega}$$

$$\phi_x(\omega) = (\phi_{x1}(\omega))^2$$




$$= \frac{(e^{j\omega - 1})^2}{-\omega^2} \quad (since \quad f_X(x) = f_{X1}(x) * f_{X1}(x))$$

$$\frac{d}{d\omega}\phi_x(\omega) = \frac{2(e^{j\omega} - 1)j(-\omega^2) - (-2\omega)(e^{j\omega} - 1)^2}{\omega^4}$$

$$= \frac{2j(e^{j\omega} - 1)}{\omega^2} + \frac{2(e^{j\omega} - 1)^2}{\omega^3}$$

Not able to evaluate at $\omega = 0$

(b) Find the expected value and variance of X.

ANS:

$$
\begin{aligned}
E[X] &= \int_0^1 x^2 dx + \int_1^2 x(2-x)dx \\
&= \frac{x^3}{3}]_0^1 + x^2]_1^2 - \frac{x^3}{3}]_1^2 \\
&= \frac{1}{3} + 3 - \frac{7}{3} \\
&= 1 \\
E[X^2] &= \int_0^1 x^3 dx + \int_1^2 x^2(2-x)dx \\
&= \frac{x^4}{4}]_0^1 + \frac{2}{3}x^3]_1^2 - \frac{x^3}{3}]_1^2 \\
&= \frac{1}{4} + \frac{14}{3} - \frac{15}{4} \\
&= \frac{14}{3} - \frac{7}{2} \\
&= \frac{7}{6} \\
\sigma_x^2 &= E[X^2] - E[X]^2 \\
&= \frac{7}{6} - 1 \\
&= \frac{1}{6}
\end{aligned}
$$

(c) Find the mean and variance of (i)Y1 = X + c

ANS:

$$
\begin{aligned}
Y_1 &= X + c \\
E[Y_1] &= E[X] + c \\
&= 1 + c \\
\sigma_{Y_1}^2 &= (1)^2\sigma_X^2 + 0 = 1 \times \frac{1}{6} = \frac{1}{6}
\end{aligned}
$$

(ii)Y2 = cX where c is a constant.
ANS:

$$Y_2 = cX$$

$$\mu_{Y2} = c\mu_X = \mu_{Y2} = c$$

$$\sigma^2_{Y2} = c^2\sigma^2_X$$

$$\sigma^2_{Y2} = c^2\sigma^2_X = \frac{c^2}{6}$$

9. Suppose X is a continuous random variable with density function

$$f_X(x) = \begin{cases} x & ; 0 \leq x \leq 1 \\ 2 - x & ; 1 \leq x \leq 2 \\ 0 & ; otherwise \end{cases}$$

Find the moment-generating function $M_X(t)$.

10. The RV X is N(5,2) and Y = 2X + 4. Find $\eta_y$,$\sigma_y$ and $f_Y(y)$.

ANS:

$$X = N(5, 2) \Rightarrow \mu_X = 5, \sigma^2_X = 2$$

$$\eta_Y = 2 * 5 + 4 = 14$$

$$E[Y^2] = E[(2X+4)^2]$$

$$= E[(4X)^2 + 16X + 16]$$

$$= 4E[X^2] + 16\mu_X + 16 = \sigma_X^2 + \mu_X^2$$

$$= 2 + 25 = 27$$

$$\sigma_Y^2 = E[Y^2] - \eta_Y^2$$

$$\Rightarrow E[X^2] = \sigma_X^2 + \mu_X^2$$

$$= 27$$

$$E[Y^2] = 4 * 27 + 16 * 5 + 16$$

$$= 204$$

$$f_Y(y) = \frac{f_X(x1)}{g'(x1)}$$

$$= \frac{1}{\sqrt{16\pi}} e^{\frac{-(y-14)^2}{16}}$$

$$\sigma_Y^2 = E[Y^2] - \mu_Y^2 204 - 196 = 8$$

$$= \sigma_Y = 2\sqrt{2}$$
$$f_Y(y) = N(14, 8)$$

11. If X is N(0,2) and Y = $3X^2$,find $\eta_y$,$\sigma_y$ and $f_Y(y)$.

ANS:

$$X = N(0, 8)$$

$$= \sigma_Y^2 = 2; \mu_X = 0$$

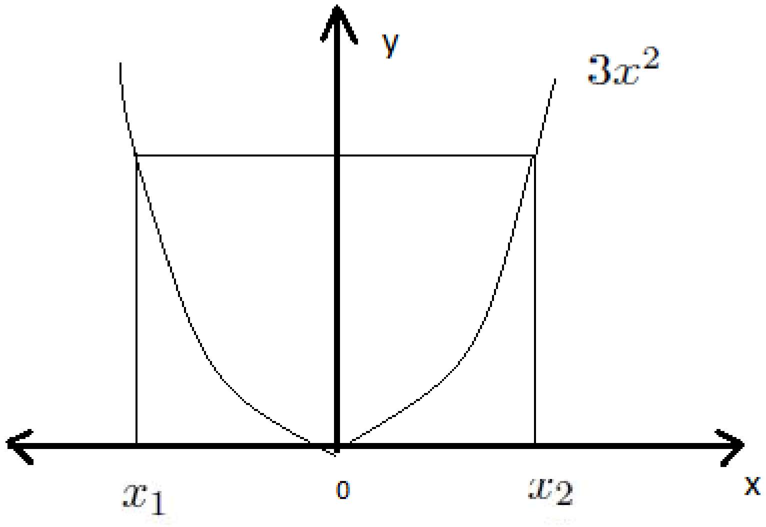
$$Y = 3X^2 = g(X) = g'(x) = 6X$$

$$x = \pm\sqrt{\frac{y}{3}}$$

$$\Rightarrow x_1 = \sqrt{\frac{y}{3}}, x_2 = -\sqrt{\frac{y}{3}}$$

$$|g'(x_1)| = 6\sqrt{\frac{y}{3}}$$

$$= 2\sqrt{3y}; |g'(x_2)| = |-6\sqrt{\frac{y}{3}}|$$




$$= 2\sqrt{3y}$$

$$f_Y(y) = \frac{f_x(x_1)}{g'(x_1)} + \frac{f_x(x_2)}{g'(x_2)}$$

$$= \frac{e^{\frac{-y}{12}}}{\sqrt{4\pi}2\sqrt{3y}} + \frac{e^{\frac{-y}{12}}}{\sqrt{4\pi}2\sqrt{3y}} \qquad since f_X(x) = \frac{1}{\sqrt{4\pi}} e^{\frac{-x^2}{2*2}} = \frac{e^{-x^2}}{4}$$

$$\Rightarrow f_Y(y) = \frac{e^{\frac{-y}{12}}}{\sqrt{12\pi y}}; y \geq 0$$

$$E[Y] = \int_0^\infty y f_Y(y) dy$$

$$= \frac{1}{\sqrt{12\pi}} \int_0^\infty y^{\frac{1}{2}} e^{\frac{-y}{12}} dy$$

$$= \frac{1}{\sqrt{12\pi}} (12)^{\frac{1}{2}} \int_0^\infty (\frac{y}{12})^{\frac{1}{2}} e^{\frac{-y}{12}} \frac{dy}{12}$$

$$= \frac{(12)^{\frac{3}{2}}}{\sqrt{12\pi}} \frac{\Gamma(\frac{3}{2})}{1}$$

$$= \frac{1}{2}\sqrt{\frac{12*12*12*\pi}{12\pi}}$$



$$= 6$$

$$E[Y^2] = \frac{1}{\sqrt{12\pi}} \int_0^\infty y^{\frac{2}{3}} e^{\frac{-y}{12}} dy$$

12. Show that

   (a) if f(x) is a Cauchy density, then $\phi(\omega) = e^{-\alpha|\omega|}$;

   ANS:

   Cauchy density $\Rightarrow f_X(x)$;

$$
\begin{aligned}
f_X(x) &= \frac{\frac{\alpha}{\pi}}{x^2+\alpha^2} \\
&= \frac{\frac{\alpha}{\pi}}{(jx+\alpha)(\alpha-jx)} \\
&= \frac{\alpha}{\pi}\left(\frac{1}{\alpha-jx}+\frac{1}{\alpha+jx}\right)\frac{1}{2\alpha} \\
taking\ FT, & \\
\phi_X(\omega) &= e^{\alpha\omega} for\ \omega < 0 + e^{-\alpha\omega}\ for\ \omega > 0 \\
&= e^{-\alpha|\omega|}
\end{aligned}
$$

   (b) if f(x) is a Laplace density, then $\phi(\omega) = \alpha^2/(\alpha^2+\omega^2)$ .

13. An RV X has a geometric distribution if

$$PX = k = pq^k \qquad k = 0, 1, .... \qquad p + q = 1$$

   Find $\Gamma(Z)$ and show that $\eta_x = \frac{q}{p}, \quad \sigma_x^2 = \frac{q}{p^2}$.

14. Let $X : N(0, \sigma^2)$.Show that

$$
\begin{aligned}
&\varepsilon = E[X^n] = 1.3....(n-1)\sigma^n \qquad && n \quad even \\
&\varepsilon = 0 && n \quad odd
\end{aligned}
$$

15. Suppose that the random variable X is described by the pdf $f_X(x) = 5x^6$; $x > 1$. What is the highest moment of X that exists?

16. An example of a Gaussian mixture PDF is

$$P_X(x) = \frac{1}{2\sqrt{2\pi}} exp[-\frac{1}{2}(x-1)^2] + \frac{1}{2\sqrt{2\pi}} exp[-\frac{1}{2}(x+1)^2]$$

   Determine the mean and variance.