1^{st}Semester M.Tech Microwave & TV Engineering and Signal Processing

Tutorial 5: Joint Distribution and Density functions

1. Suppose that X and Y are independent continuous Random Variables having densities $f_X(x)$ and $f_Y(y)$respectively.
   Show that$P(X < Y) = \int_{-\infty}^{\infty} F_X(y)\, F_Y(y)\, dy$

   Solution:

   $P(X,Y) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x,y)\, dy\, dx$

   $= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_X(x) f_Y(y)\, dx\, dy$

   $= \int_{-\infty}^{\infty} [\int_{-\infty}^{y} f_X(x)\, dx]\, dy$

   $= \int_{-\infty}^{\infty} F_X(y) f_Y(y)\, dy$

2. Given the probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{\pi}(1 - x^2 - y^2) & for \quad 0 < x^2 + y^2 < 1 \\ 0 & otherwise \end{cases}$$

   Solution:

   $P(X^2 + Y^2 < B^2) = \int \int F_{XY}(x,y)\, dx\, dy$

   Put

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$dx\, dy = r dr d\theta$$

$P(X^2+Y^2<B^2) = \int_{r=0}^{b}\int_{\theta=0}^{2\pi} \frac{2}{\pi}(1-r^2)r\,dr\,d\theta$

$$= 4\int_{r=0}^{b}(1-r^2)\,r\,dr$$

$$= 4\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^b$$

$$= 2b^2 - b^4$$

3. A smooth surface table is ruled with equidistant parallel lines at distance D apart.A needle of length L where $L \le D$ is randomly dropped on this table.What is the probability that the needle will intersect one of them?

Solution:

$\theta \longrightarrow$ angle made by the needle with line which is perpendicular to the line.

$XY \longrightarrow$ RV which is the perpendicular distance from the centre of the needle to the line.

$XU(0,D/2), \theta U(0,\pi/2) \longrightarrow$ both are independent

$$f_{X\theta}(X,\theta) = \begin{cases} \frac{4}{\pi} & ;0<x<D/2, 0<\theta<\pi/2 \\ 0 & ;otherwise \end{cases}$$

If needle intersects a line , X should be$\le \frac{L}{2}\cos\theta, \theta$should be between $0, \pi/2$
P(needle intersects a line)$= P(X <= \frac{L}{2}\cos\theta\,, 0<\theta<\frac{\pi}{2})$

$$= \int_{x=0}^{\frac{L}{2}\cos\theta}\int_{\theta=0}^{\pi/2} \frac{4}{D\pi}\,d\theta\,dx$$

$$= \int_0^{\pi/2} \frac{4}{\pi}\frac{L}{2}\cos\theta\,d\theta$$

$$= \frac{2L}{D\pi}\left[\sin\theta\right]_0^{\pi/2}$$

$$= \frac{2L}{D\pi}$$

4. Let X and Y be continuous random variable with joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-y} & ;0<x<y \\ o & ;otherwise \end{cases}$$

Solution:

$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$

$= \int_{x}^{\infty} e^{-y}\,dy$

$= \left[\frac{e^{-y}}{-1}\right]_x^{\infty}$

$= \frac{0-e^{-x}}{-1} = e^{-x} \quad ; x \geq 0$

$f_Y(y) = \int_{x=0}^{y} f_{XY}(x,y)\,dx$

$= \int_0^y e^{-y}\,dx = ye^{-y} \quad ; y \geq 0$

5. Suppose (X,Y)is evenly distributed over the area bounded by

$$y = x^2 \quad and \quad y = 4$$

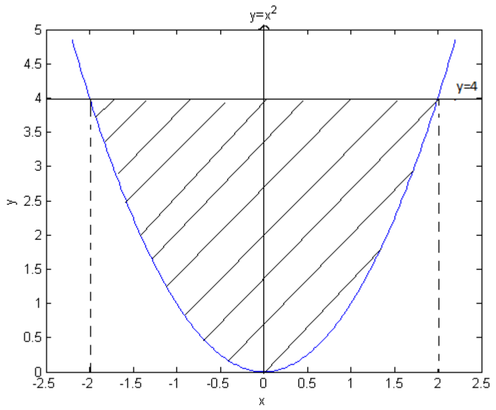
a)Find the joint density function of X and Y
b)Find $P(X < 0, Y < 3)$

Solution:

(X, Y) is evenly distributed
$\implies$ height is constant=k

$f_{XY}(x,y) = k$

a)



$\int_{x=-2}^{2} \int_{y=x^2}^{4} k = 1$

$\implies k \int_{-2}^{2} [y]_{x^2}^{y}\; dx = 1$

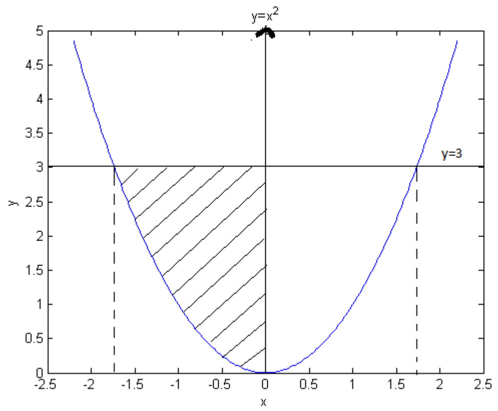
$k\int_{-2}^{2}(A - x^2)\,dx = 1$

$\Longrightarrow k\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 1$

$\Longrightarrow k = \frac{3}{32}$

$f_{XY}(x,y) = \frac{3}{32}$

b)



$P(X < 0, Y < 3) = \int_{x=-\sqrt{3}}^{0}\int_{y=x^2}^{3}\frac{3}{32}\,dy\,dx$

$= \frac{3}{32}\int_{-\sqrt{3}}^{0}[y]_{x^2}^{3}\ dx \quad = \quad \frac{3}{32}\int_{-\sqrt{3}}^{0}(3 - x^2)\,dx$

$= \frac{3}{32}\left[3x - \frac{X^3}{3}\right]_{-\sqrt{3}}^{0}$

$= \frac{3\sqrt{3}}{16}$

6. Consider an experiment drawing randomly 3 balls from an urn containing 2 red,3 white,and 4 blue balls.Let (X, Y) be bivariate random variable where X, Y denote respectively the number of red and white balls chosen.

   (a) Find the range of X and Y.
   (b) Find joint probability mass function of (X, Y)

Solution:

(a) X varies from 0 to 2 ; $X=\{0, 1, 2\}$

Y varies from 0 to 3-X ; Y$=\{0, 1 \cdots 3 - X\}$

(b) P(X=0,Y=0)= $\frac{4C_3}{9C_3} \times 3C_0$ P(X=1,Y=0)=$\frac{2C_1 \times 4C_2 \times 3C_0}{9C_3}$

P(X=0,Y=1)= $\frac{3C_1}{9C_3} \times 4C_2$ P(X=1,Y=1)=$\frac{2C_1 \times 4C_1 \times 3C_1}{9C_3}$

P(X=0,Y=2)= $\frac{4C_1 \times 3C_2}{9C_3}$ P(X=1,Y=2)=$\frac{2C_1 \times 4C_0 \times 3C_2}{9C_3}$

P(X=0,Y=3)= $\frac{4C_0}{9C_3} \times 3C_3$ P(X=1,Y=3)=0

In general,P$(X = x_i, Y = y_j) = \frac{2C_i \times 3C_j \times 4C_{[}3-(i+j)]}{9C_3}$ $, 0 < i < 2; 0 < j < (3 - i)$

7. Consider a bivariate random variable (X, Y) where X,Y denote horizontal and vertical miss distances from a target when a bullet is fired.Assuming X, Y are independent and that probability of bullet landing on any point of XY plane depends only on distance of point from target.Show that (X, Y) is a bivariate normal random variable.

Solution:

Probability of bullet landing on any point of XY plane depend only on distance of point from target.
i.e it depends only on $r = \sqrt{x^2 + y^2}$
where

X : RV→ horizontal miss distance

Y : RV → vertical miss distance vertical miss distance
pdf of miss distance,

$f_{XY}(x, y) = f_X(x).f_Y(y)$ where X and Y are independent.

Also $f_{XY}(x, y) = g(\sqrt{x^2 + y^2}) = g(r)$

$\Rightarrow g(r) = f_X(x).f_Y(y)$

$\frac{\partial g(r)}{\partial r} = \frac{dg(r)}{dr}.\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$

$\Rightarrow \frac{\partial g(r)}{\partial x} = g'(r)\frac{x}{r}$

So, $\frac{x}{r}g'(r) = f_X^{-1}(x)f_Y(y)$; (Dividing both sides by $xg(r)$)

$$\frac{1}{r}\frac{g'(r)}{g(r)} = \frac{1}{x}\frac{f'_X(x)}{f_X(x)} \tag{1}$$

The RHS of (1) is independent of y,and LHS is a function of $r = \sqrt{x^2 + y^2}$

$\Rightarrow$ both sides are independent of x and y.Hence,

$\frac{1}{r}\frac{g'(r)}{g(r)} = \alpha = constant \Rightarrow \frac{g'(r)}{g(r)} = \alpha r$

Integrating both sides, $ln\{g(r)\} = \frac{\alpha r^2}{2} + c$ so $g(r) = g\left(\sqrt{x^2+y^2}\right) = f_{XY}(x,y) = Ae^{\frac{\alpha\left(x^2+y^2\right)}{2}}$

$\Rightarrow X$ and Y are normal with zero mean and variance $\sigma^2 = \frac{-1}{\alpha}$

8. The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & ; \quad 0<x<1, 0<y<1 \\ 0 & \quad Otherwise \end{cases}$$

Compute the conditional density of X,given that $X = Y$,where $0 < y < 1$.

Solution:

$f_{X|Y}(x|y) = \frac{f_{XY}(x,y}{f_Y(y)}$

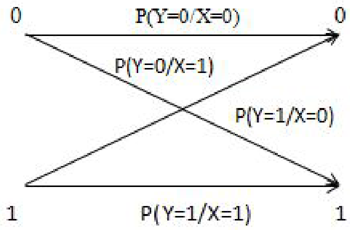
$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_0^1 \frac{12}{5}x(2-x-y)dx \\ &= \frac{12}{5}\left(\frac{2}{3} - \frac{y}{2}\right) \\ &= \frac{(8-6y)}{5} \end{aligned}$$

$\Rightarrow f_Y(y)|_{y=x} = \frac{(8-6x)}{5}$

$$\begin{aligned} f_{XY}(x,x) &= \frac{12}{5}x(2-2x) \\ &= \frac{24x(1-x)}{5} \end{aligned}$$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{24x(1-x)}{(8-6x)} \\ &= \frac{12x(1-x)}{(4-3x)} \end{aligned}$$

9. Consider a binary communication channel in figure.



Let X, Y be bivariate random variables,where X is the input to the channel and Y is the output of the channel.Let $P(X = 0) = 0.5, P(Y = 1|X = 0) = 0.1$ and $P(Y = 0|X = 1) = 0.2$

(a) Find joint probability matrix of (X, Y)

(b) Find the marginal densities of X, Y

Solution:

P(X=0)=0.5=P(X=1);

$P(Y = 1|X = 0) = 0.1;$ $P(Y = 0|X = 1) = 0.2$

$\Rightarrow P(Y = 1, X = 0) = 0.1 \times 0.5 = 0.05;$ $\Rightarrow P(Y = 0, X = 1) = 0.2 \times 0.5 = 0.1$

(a) Joint probability matrix of (X, Y)

X \ Y	0	1
0	0.45	0.05
1	0.1	0.4

(b)
$$
\begin{aligned}
P(X = 0) &= 0.5 \\
P(X = 1) &= 0.5 \\
P(Y = 0) &= 0.55 \\
P(Y = 1) &= 0.45
\end{aligned}
$$

(c) $P(Y=0,X=1) \;=\; 0.1 \;\neq\; P(Y=0)P(X=1)$
$\Rightarrow X$ and Y are not independent.

10. Two fair dice are thrown.Let X=0 or 1 according to whether the first die shows an even number or an odd number.Similarly,let Y=0 or 1 according to the second die.Find the joint probability mass function of X and Y.

Solution:

D_1 is odd $\Rightarrow X=1$ $\qquad$ D_2 is odd $\Rightarrow Y=1$

D_1 is even $\Rightarrow X=0$ $\qquad$ D_2 is even $\Rightarrow Y=0$

$\mathrm{P}(X=1|Y=0)=0.5 \;\Rightarrow \mathrm{P}(X=1|Y=0)=0.5\times 0.5=0.25$

Similarly,for all the 4 cases.

So, $\mathrm{P}(X=x_i,Y=y_j)=0.25$; $\qquad X=\{0,1\}\,,Y=\{0,1\}$

11. Let the probability density function of X_1 and X_2 be given by

$$f_{X_1,X_2}(x_1,x_2)=\begin{cases} e^{-(x_1+x_2)} & for \quad x_1>0,x_2>0 \\ 0 & otherwise \end{cases}$$

Consider two random variables Y_1 and $Y_2, Y_2=\frac{X_1}{X_1+X_2}$. Find the joint density of Y_1 and Y_2 and marginal density of Y_2.

Solution:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x).f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)dx} \\ &= \frac{\frac{1}{2\sqrt{2\pi\sigma^2}}e^{\frac{-(y-x)^2}{2\sigma^2}}}{\int_{-1}^{1}\frac{1}{2\sqrt{2\pi\sigma^2}}e^{\frac{-(y-x)^2}{2\sigma^2}}dx} \\ &= \frac{exp\left\{-\frac{(y-x)^2}{2\sigma^2}\right\}}{\int_{-1}^{1} exp\left\{-\frac{(y-x)^2}{2\sigma^2}\right\}dx} \end{aligned}$$

Put

$$u=\frac{(y-x)}{\sigma}$$

$$du=\frac{-dx}{\sigma}$$

$$x = -1 \Rightarrow u = \frac{(y+1)}{\sigma}$$
$$x = 1 \Rightarrow u = \frac{(y-1)}{\sigma}$$

So,

$$f_{X|Y}(x|y) = \frac{exp\left\{-\frac{(y-x)^2}{2\sigma^2}\right\}}{\sqrt{2\pi\sigma^2}\left\{G\left(\frac{y+1}{\sigma}\right) - G\left(\frac{y-1}{\sigma}\right)\right\}}$$