1st Semester M.Tech Microwave & TV Engineering And Signal Processing

Tutorial 6: Functions of Two Random Variables

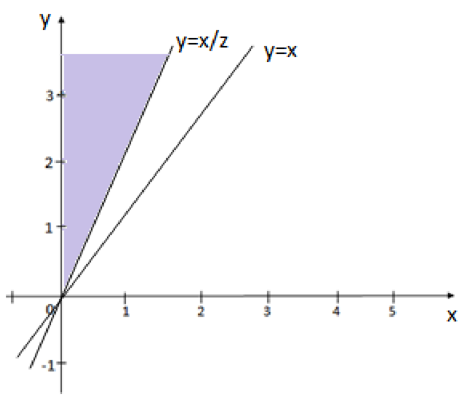
1. Let X and Y be independent exponential random variables with common parameter λ. Find $f_Z(z)$ by defining $Z = \dfrac{min(x;y)}{max(x;y)}$

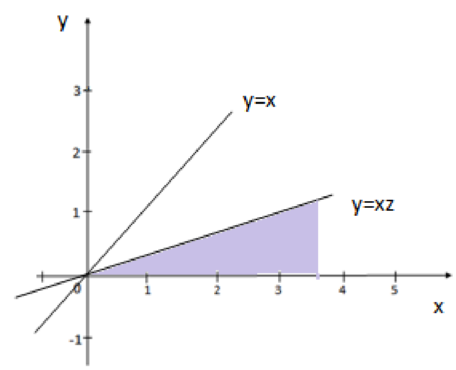
Solution:

$$Z = \frac{min(X,Y)}{max(X,Y)} = \begin{cases} \dfrac{X}{Y};\ X \le Y \\ \dfrac{Y}{X};\ X > Y \end{cases}$$

$$F_Z(z) = P(\frac{X}{Y} \le z, X \le Y) + P(\frac{Y}{X} \le z, X > Y)$$

$$Z = \frac{min}{max} \longrightarrow -1 \le z \le 1$$





Since X and Y are exponential,x>0,y>0.So $0 \le z \le 1$

The area is $\{Y > 0\} \cap \{Y \ge X\} \cap \{Y \ge \dfrac{X}{Z}\}$

This area is $\{X > 0\} \cap \{X > Y\} \cap \{Y \ge XZ\}$

$$F_Z(z) = \int_{y=0}^{\infty} \int_{x=0}^{yz} f_{XY}(x,y)\ dxdy + \int_{x=0}^{\infty} \int_{y=0}^{xz} f_{XY}(x,y)\ dydx$$

$$= \int_0^{\infty} F_{XY}(x,xz)dx$$

$$f_Z(z) = \int_0^{\infty} y f_{XY}(yz,y)dy + \int_0^{\infty} x f_{XY}(x,xz)dx$$

$$f_{XY}(x,y) = \lambda^2 e^{-\lambda(x+y)}$$

$$F_Z(z) = \int_{y=0}^{\infty}\int_{x=0}^{yz} \lambda^2 e^{-\lambda(x+y)} dxdy + \int_{x=0}^{\infty}\int_{y=0}^{xz} \lambda^2 e^{-\lambda(x+y)} dydx$$
$$= \int_{y=0}^{\infty} \lambda e^{-\lambda x} dx \int_{x=0}^{yz} \lambda e^{-\lambda x} dy + \int_{x=0}^{\infty} \lambda e^{-\lambda x} dy \int_{y=0}^{xz} \lambda e^{-\lambda x} dx$$
$$= \int_{y=0}^{\infty} \lambda e^{-\lambda y}(1 - e^{-\lambda yz}) dy + \int_{x=0}^{\infty} \lambda e^{-\lambda x}(1 - e^{-\lambda xz} dx$$
$$= \int_{0}^{\infty} \lambda e^{-\lambda yz} dy + \int_{y=0}^{\infty} \lambda e^{-\lambda y(1+z)} dy + \int_{x=0}^{\infty} \lambda e^{-\lambda x} dx - \int_{x=0}^{\infty} \lambda e^{-\lambda x(z+1)} dx$$
$$= 1 - \frac{1}{(1+z)} + 1 - \frac{1}{(1+z)} = 2 - \frac{2}{(1+z)}$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{2}{(1+z)^2}; \quad 0 < z < 1$$

2. Let X $\sim$ U(0; 1) and Y $\sim$ U(0; 1) be independent random variables. Find the density function of Z, defining $Z = \sqrt{-2lnX}cos(2\pi Y)$

Solution:

X $\sim$ u(0,1),Y $\sim$ u(0,1)

Z=$\sqrt{-2lnX}cos(2\pi Y)$; W=Y

Z=$\sqrt{-2lnX}cos(2\pi W)$;

Range of $Z \longrightarrow cos(2\pi Y)$ from $-1\ to + 1$

$\sqrt{ln\frac{1}{X^2}} as\ X\ from\ 0\ to 1 \rightarrow \infty\ to\ 0$

$So\ Z\ from - \infty\ to\ \infty$

$$J^Y(x,y) = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial W} \end{vmatrix} = \begin{vmatrix} -zsec^2(2\pi W)\ e^{-\frac{-z^2 sec^2(2\pi W)}{2}} & \frac{\partial X}{\partial W} \\ 0 & 1 \end{vmatrix}$$
$$= -zsec^2(2\pi W)e^{\frac{-z^2 sec^2(2\pi W)}{2}}$$

$X = e^{-\frac{zsec(2\pi W)^2}{2}}; \quad Y = W$

$f_{ZW}(z,w) = f_x(x_1)f_y(y_1)|J^{-1}| = zsec^2(2\pi W)e^{\frac{-z^2 sec^2(2\pi W)}{2}}$

$$f_Z(z) = \int_{w=0}^{1} zsec^2(2\pi W)e^{\frac{-z^2 sec^2(2\pi W)}{2}}\ dw$$
$$= \int_{w=0}^{1} zsec^2(2\pi W)e^{\frac{-z^2(1+tan^2(2\pi W))}{2}}\ dw$$

Put
u=z tan(2 π W)
du= $2\pi z sec^2(2\pi W)d$
w= 0 $\longrightarrow$u= 0
w= 1 $\longrightarrow$u= 0
w= $\frac{1}{4}$ $\longrightarrow$u= ∞
w= $\frac{3}{4}$ $\longrightarrow$u= $-\infty$

Hence

$$\begin{aligned} f_Z(z) &= \int_{u=-\infty}^{\infty} \frac{1}{2\pi} e^{(\frac{-z^2}{2}+\frac{-u^2}{2})}\ du \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{\infty} e^{\frac{-u^2}{2}}\ du \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}};\ -\infty < z < \infty \end{aligned}$$

3. Let X and Y be independent gamma random variables, X $\sim$ G(m;α) and Y $\sim$ G(n;α). Define Z = X + Y and W = X/Y . Show that Z and W are independent random variables.

Solution:
$X \sim G(m,\alpha); Y \sim G(n,\alpha)$

$Z = X + Y,\ W = \frac{X}{Y}$

$f_X(x) = \frac{x^{m-1}e^{\frac{-x}{2}}}{\Gamma(m)\alpha^m}; x > 0$

$f_Y(y) = \frac{y^{n-1}e^{\frac{-y}{2}}}{\Gamma(n)\alpha^n}; y > 0$

$J(x,y) = \begin{vmatrix} \frac{\partial Z}{\partial X} & \frac{\partial Z}{\partial Y} \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-1}{y} & \frac{-x}{y^2} \end{vmatrix} = \frac{-x}{y^2} - \frac{1}{y} = \frac{x+y}{y^2}$

$z = x + y;\quad x = wy; \Longrightarrow z = (w+1)y \Longrightarrow y = \frac{z}{w+1}; \frac{zw}{w+1}$

$|J(x_1,y_1)| = \frac{|-z|}{\frac{|z|}{|w+1|}} = \frac{z(w+1)^2}{z^2} = \frac{(w+1)^2}{z}$

$$f_{ZW}(z,w) = \frac{f_{XY}(x_1,y_1)}{|J(x_1,y_1)|} = \frac{f_X(x_1).f_Y(y_1)}{|J(x_1,y_1)|}$$

$$= \frac{x^{m-1}e^{-\frac{x}{\alpha}}y^{n-1}e^{\frac{-y}{\alpha}}}{\Gamma(m)\alpha^m\Gamma(n)\alpha^n}.\frac{z}{(w+1)^2}$$

$$= \frac{\left(\frac{zw}{1+w}\right)^{m-1}\left(\frac{z}{1+w}\right)^{n-1}e^{-\frac{(x+y)}{\alpha}}z}{\Gamma(m)\Gamma(n)\alpha^{m+n}(w+1)^2}$$

$$= \frac{z^{(m+n-1)}w^{(m-1)}e^{-\frac{y}{\alpha}}}{\Gamma(m)\Gamma(n)\alpha^{(m+n)}(w+1)^{m+n}}$$

$$= \underbrace{\frac{z^{(m+n-1)}e^{\frac{-z}{\alpha}}}{\Gamma(m+n)\alpha(m+n)}}_{Function\ of\ z}.\underbrace{\frac{\Gamma(m+n)w^{(m-1)}}{\Gamma(n)\Gamma(m)(1+w)^{(m+n)}}}_{Function\ of\ w=G(w)}$$

$$W = \frac{X}{Y} \Longrightarrow \qquad f_W(w) = \int_{-\infty}^{+\infty}|y|f_X(yw)f_Y(y)\ dy$$

$$= \int_0^\infty \frac{y(yw)^{(m-1)}e^{-y\frac{w}{\alpha}}}{\Gamma(m)\alpha^m}\frac{y^{(n-1)}e^{-\frac{y}{\alpha}}}{\Gamma(n)\alpha^n}\ dy$$

$$= \int_0^\infty \frac{y^{(m+n-1)}w^{(m-1)}e^{-\frac{y(w+1)}{\alpha}}}{\Gamma(m)\Gamma(n)\alpha^{(m+n)}}\ dy$$

$$= \frac{w^{(m-1)}}{\Gamma(n)\Gamma(m)\alpha^{(m+n)}}.\int_0^\infty y^{(m+n-1)}e^{-\frac{y(w+1)}{\alpha}}\ dy$$

Let $u = \frac{y(w+1)}{\alpha}$

Then $du = \frac{(w+1)}{\alpha}\ dy$

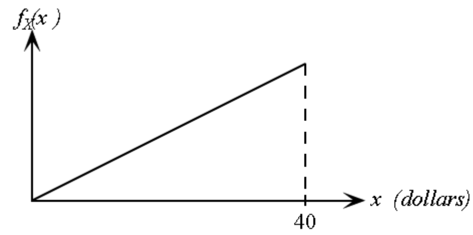
Therefore, $f_W(w) = \frac{w^{(m-1)}\alpha^{(m+n-1)}\alpha}{\Gamma(m)\Gamma(n)\alpha^{(m+n)}(w+1)^{(m+n-1)}(w+1)}\int_0^\infty u^{(m+n-1)}e^{-u}\ du$

$$= \frac{w^{(m-1)}\Gamma(m+n)}{\Gamma(m)\Gamma(n)(1+w)^{(m+n)}} = G(w)$$

Therefore $f_Z(z) = \frac{z^{(m+n-1)}e^{-\frac{z}{d}}}{\Gamma(m+n)\alpha^{(m+n)}}$

$\Longrightarrow f_{ZW}(z,w) = f_Z(z)f_W(w) \Longrightarrow$ Z and W are independent.

4. Paul is vacationing in Monte Carlo. The amount X he takes to the casino each evening is a random variable with the PDF shown in Fig.1. At the end of each night, the amount Y that he has on leaving the casino is uniformly distributed between zero and twice the amount he took in



a. Determine the joint PDF f_{XY} (x,y).

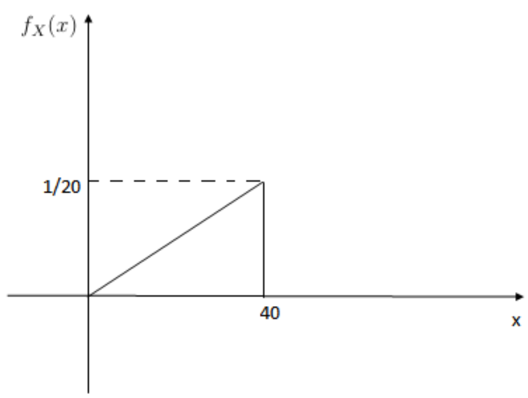
b. What is the probability that on any given night Paul makes a profit at the casino? Justify.

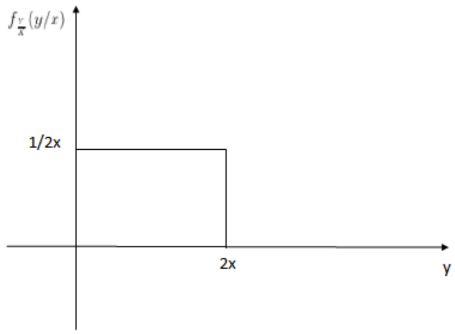
c. Find and sketch the probability density function of Paul's profit on any particular night, Z = Y-X.

Solution:

$$f_X(x) = \begin{cases} \frac{1}{800}x; 0 \le x \le 40 \\ 0;\text{elsewhere} \end{cases} \qquad f_{\frac{Y}{X}}(y/x) = \begin{cases} \frac{1}{2x}; 0 \le y \le 2x \\ 0;\text{elsewhere} \end{cases}$$

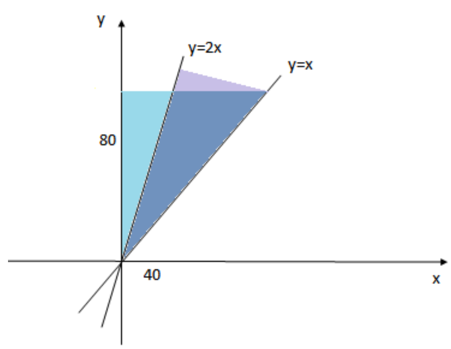
a) $f_{XY}(x,y) = f_{\frac{Y}{X}} f_{X}(x) = \frac{1}{800}.x.\frac{1}{2x} = \frac{1}{1600};\qquad 0 \le x \le 40, 0 \le y \le 2x;$



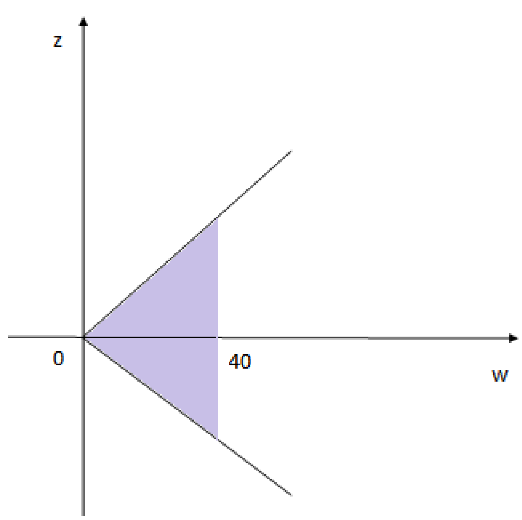


b) $P(Y \geq X) = \int_{x=0}^{40} \int_{y=x}^{2x} f_{XY}(x,y)\; dydx$

$$= \frac{1}{1600} \int_0^{40} (2x - x)dx$$
$$= \frac{1}{1600} . \frac{x^2}{2}]_0^{40}$$
$$= \frac{1}{2}$$



c) Z=Y-X,W=X



$$J(x,y) = \begin{vmatrix} \dfrac{\partial Z}{\partial X} & \dfrac{\partial Z}{\partial Y} \\ \\ \dfrac{\partial W}{\partial X} & \dfrac{\partial W}{\partial Y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$x = w; \quad y = z + x = z + w$
$0 \le x \le 40 \longrightarrow 0 < w < 40$
$0 \le y \le 2x \longrightarrow -x < w < 40$
$0 \le y \le 2x \longrightarrow -x < z < x$

$$\int_w f_{ZW}(z,w)\ dw = \underbrace{\int_{w=z}^{40} \frac{1}{1600}\ dw}_{(z>0)} + \underbrace{\int_{w=-z}^{40} \frac{1}{1600}\ dw}_{(z<0)} = \frac{1}{1600}(40-z) + \frac{1}{1600}(40+z)$$

$$= \frac{1}{1600}(40 - |z|); \qquad -x < z < x$$

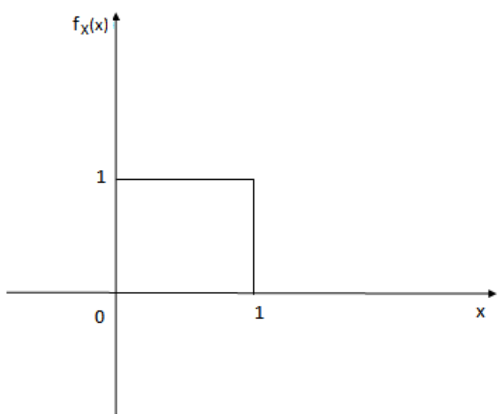
5. X and Y are independent uniformly distributed random variables in (0; 1). Let W = $max(X,Y)$ and Z = $min(X,Y)$. Find the pdf of

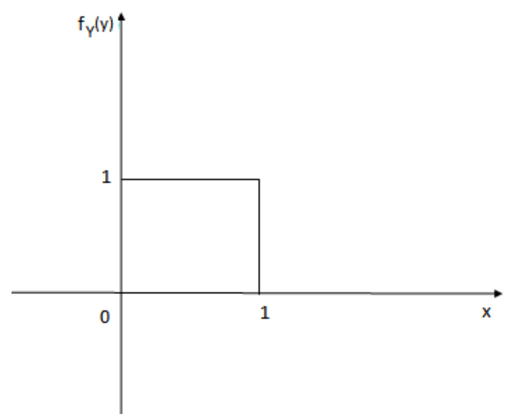
   a. r=w-z

   b. s=w+z

Solution:
X u(0,1);Y u(0,1);W max(X,Y);Z=min(X,Y)

a) r=w−z





$R =$

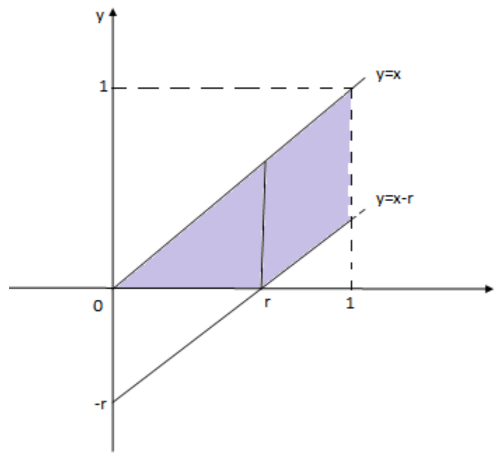
$W - Z = max(X,Y) - min(X,Y)$

$= \begin{cases} X - Y; & X \le Y \\ Y - X; & X < Y \end{cases} \quad F_R(r) = P(X - Y < r, X \ge Y) + P(Y - X < r, X < Y)$

$x \longrightarrow 0 \longrightarrow 0\ to\ r \Longrightarrow F_R(r) = \dfrac{r^2}{2}$

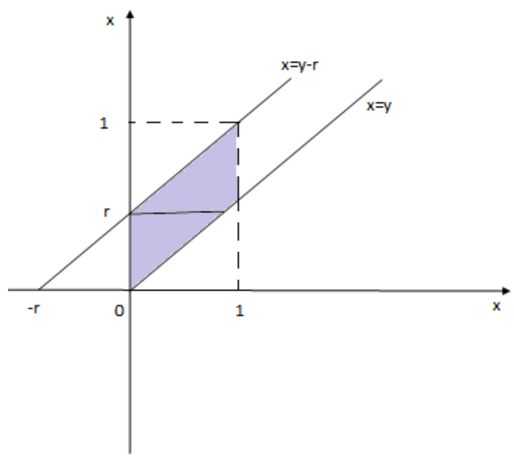
$x \longrightarrow r\ to\ 1 \Longrightarrow F_R(r) = \int_{x=r}^{1} \int_{y=x-r}^{x} 1.dydx$

$= \int_r^1 rdx = rx]_r^1 = r(1-r)$



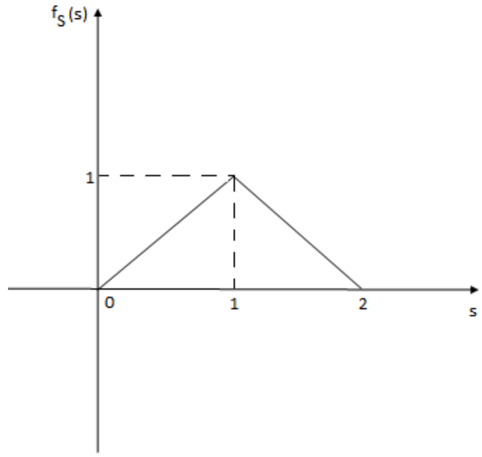
$F_R(r) = r^2 + 2r(1-r) = 2r - r^2$

$$f_R(r) = \begin{cases} 2(1-r);\ 0 < r < 1 \\ 0; otherwise \end{cases}$$



b) $S = W + Z$
$= max(X, Y) + min(X, Y)$
$= X + Y$

$f_S(s) = f_X(x) * f_Y(y)$
$$= \begin{cases} s;\ 0 < s < 1 \\ 2 - s;\ 1 < s < 2 \end{cases}$$



6. Consider independent Gaussian random variables X and Y with the joint probability density function $f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}}$. Random variables Z and W are defined in terms of X and Y by the transformations Z = $\sqrt{X^2+Y^2}$ and W = Y/X. Find F_{ZW}, F_Z and F_W.

Solution:
$Z = \sqrt{X^2+Y^2}$; $W = \frac{Y}{X}$

$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2}$; $-\infty < x, y < \infty$

Range of z = 0 to ∞
Range of w = $-\infty$ to ∞
Taking polar co-ordinates,

$$\left.\begin{matrix} X = R\cos\theta \\ Y = R\sin\theta \end{matrix}\right\} \implies Z = R\ ,\ \theta = \tan^{-1} W$$

$F_{ZW}(z,w) = \iint_{x^2+y^2 \le z^2} f_{xy}(x,y)dxdy$

Over this area r can vary from 0 to z.

$$for\ x \ge 0, \omega\ varies\ from\ -\infty\ to\ \omega \Rightarrow \theta\ from\ -\frac{\pi}{2}\ to\ \tan^{-1} w$$
$$x < 0, \omega\ varies\ from\ -\infty\ to\ \omega \Rightarrow \theta\ from\ \pi - \frac{\pi}{2}\ to\ \pi + \tan^{-1} w$$

$$\begin{aligned} F_{ZW}(z,w) &= \int_{r=0}^{z} \int_{\theta=-\frac{\pi}{2}}^{tan^{-1}\omega} f_{xy}(x,y)dxdy + \int_{r=0}^{z} \int_{\theta=\frac{\pi}{2}}^{\pi+tan^{-1}\omega} f_{r\theta}(r,\theta) rdrd\theta \\ &= 2\int_{r=0}^{z} \int_{\theta=-\frac{\pi}{2}}^{tan^{-1}\omega} \frac{1}{2\pi^2} e^{-\frac{r^2}{2\sigma^2}} rdrd\theta \\ &= 2(tan^{-1}\omega + \frac{\pi}{2}) \int_{0}^{z} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} rdr \end{aligned}$$

Let u = $\frac{r^2}{2\sigma^2}$
$du = \frac{r}{\sigma^2} dr$

$$F_{ZW}(z,w) = (tan^{-1}\omega + \frac{\pi}{2})\frac{1}{\pi}\int_0^{\frac{z^2}{2\sigma^2}} e^{-u}du$$
$$= (tan^{-1}\omega + \frac{\pi}{2})\frac{1}{\pi}(1 - e^{-\frac{z^2}{2\sigma^2}}) \quad ;0 < z < \infty \text{ and } -\infty < \omega < \infty$$
$$F_Z(z) = 1 - e^{-\frac{z^2}{2\sigma^2}}u(z)$$
$$F_Z(z) = \frac{1}{2} + \frac{1}{\pi}tan^{-1}w \ ; -\infty < \omega < \infty$$

7. Let θ be a prescribed angle and consider the rotational transformation V = X$\cos\theta$ + Y $\sin\theta$ and W = X$\sin\theta$ - Y $\cos\theta$ with $f_{XY}(x,y) = \frac{1}{2\pi\sigma^2}e^{\frac{-(x^2+y^2)}{2\sigma^2}}$. Find $f_{VW}(v,w)$.

Solution: $f_{VW} = \frac{f_{XY}}{|J|}$

$$J = \begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = -1$$

$$|J| = 1$$
$$v = x\cos\theta + y\sin\theta$$
$$w = x\sin\theta - y\cos\theta$$
$$f_{VW}(v,w) = \frac{f_{XY}(x_1,y_1)}{1}$$

Solving for x_1 and y_1

$$x_1 = \frac{v - y_1\sin\theta}{\cos\theta};$$
$$x_1 = \frac{w + y_1\cos\theta}{\sin\theta}$$

Equating ,

$$\frac{v - y_1\sin\theta}{\cos\theta} = \frac{w + y_1\cos\theta}{\sin\theta}$$
$$v\sin\theta - y_1\sin^2\theta = w\cos\theta + y_1\cos^2\theta \Longrightarrow v\sin\theta - w\cos\theta = y_1$$
$$y_1 = \frac{v - x_1\cos\theta}{\sin\theta};$$
$$y_1 = \frac{x_1\sin\theta - w}{\cos\theta}$$

Equating ,

$$v\cos\theta - x_1\cos^2\theta = x_1\sin^2\theta - w\sin\theta \Longrightarrow x_1 = v\cos\theta + w\sin\theta$$

$$\therefore f_{VW}(v,w) = f_{XY}(v\cos\theta + w\sin\theta, v\sin\theta - w\cos\theta)$$
$$= \frac{1}{2\pi\sigma^2}exp - (v^2\cos^2\theta + w^2\sin^2\theta$$
$$+ 2vw\sin\theta\cos\theta + \frac{v^2\sin^2\theta + w^2\cos^2\theta}{2\sigma^2} - 2vw\sin\theta\cos\theta)$$
$$= \frac{1}{2\pi\sigma^2}e^{-\left(\frac{v^2+w^2}{2\sigma^2}\right)}; -\infty < v,w < \infty$$

8. Let X and Y be independent identically distributed exponential random variables with common parameter λ. Find the pdf of

   a. Z = Y/max(X,Y)
   b. W = X/min(X,2Y)

Solution:

a. $Z = \dfrac{Y}{max(X,Y)} = \begin{cases} \dfrac{Y}{X} & ; X \geq Y \\ 1 & ; X < Y \end{cases} \quad \Rightarrow 0 \leq z \leq 1$

$F_Z(z) = P(Z \leq z) = P(\dfrac{Y}{X} \leq z, X \geq Y)$

$$\begin{aligned} f_Z(z) &= \int_{x=-\infty}^{\infty} |x| f_X(x) f_Y(xz) dx \\ &= \int_{x=0}^{\infty} x f_X(x) f_Y(xz) dx \\ &= \int_{x=0}^{\infty} x \lambda e^{-\lambda x} \lambda e^{-\lambda x z} dx \\ &= \lambda^2 \int_{x=0}^{\infty} x e^{-\lambda x(1+z)} dx \\ &= \lambda^2 \left\{ \frac{x e^{-\lambda x(1+z)}}{-\lambda(1+z)} - \int_0^{\infty} \frac{e^{-\lambda x(1+z)}}{-\lambda(1+z)} \right\} \\ &= \frac{\lambda^2}{\lambda(1+z)} \left[\left\{ 0 - \lim_{x \to \infty} \frac{x}{e^{\lambda x(1+z)}} \right\} + \frac{e^{\lambda x(1+z)}}{-\lambda(1+z)} \right]_0^{\infty} \\ &= \frac{\lambda}{(1+z)} \left\{ 0 - \frac{1}{\lambda(1+z)}(0-1) \right\} \\ &= \frac{1}{(1+z)^2} \qquad ; 0 \leq z < 1 \end{aligned}$$

$$\begin{aligned} P(Z=1) = P(X<Y) &= \int_{y=0}^{\infty} \int_{x=y}^{\infty} \lambda^2 e^{-\lambda(x+y)} dx dy = \lambda^2 \int_0^{\infty} e^{-\lambda y} \left. \frac{e^{-\lambda y}}{-\lambda} \right]_y^{\infty} dy \\ &= \lambda \int_0^{\infty} e^{-\lambda y} e^{-\lambda y} dy \\ &= \lambda \int_0^{\infty} e^{-2\lambda y} dy \\ &= \lambda \left. \frac{e^{-2\lambda y}}{-2\lambda} \right]_0^{\infty} \\ &= \frac{-1}{2}(0-1) = \frac{1}{2} \end{aligned}$$

b. $W = \dfrac{X}{min(X,2Y)}$

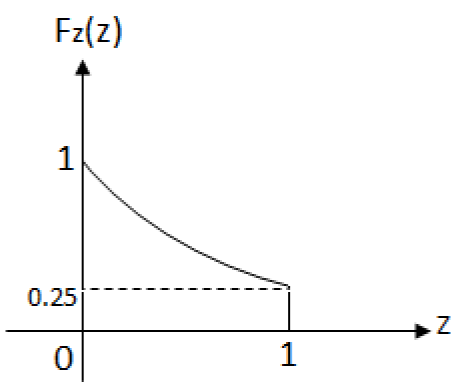
$= \begin{cases} \dfrac{X}{2Y} & ; X \geq 2Y \\ 1 & ; X < 2Y \end{cases} \qquad 1 \leq w < \infty$

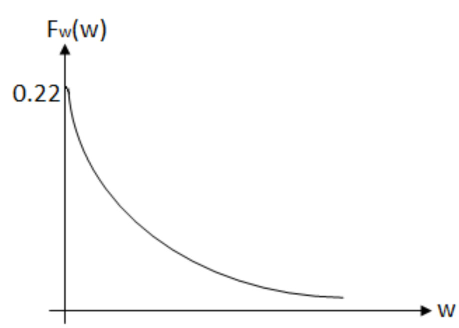
$$
\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} |2y| f_X(2yw) f_Y(y) dy \\
&= \int_0^{\infty} 2y\lambda e^{-2\lambda yw} \lambda e^{-\lambda y} dy \\
&= 2\lambda^2 \int_0^{\infty} y e^{-\lambda y(1+2w)} dy \qquad \text{y is exponential}
\end{aligned}
$$

Let u=λ y(1+2w)
du= λ (1+2w)dy
y=0 $\Rightarrow$ u=0
y= $\infty \Rightarrow$ u= ∞.
Therefore

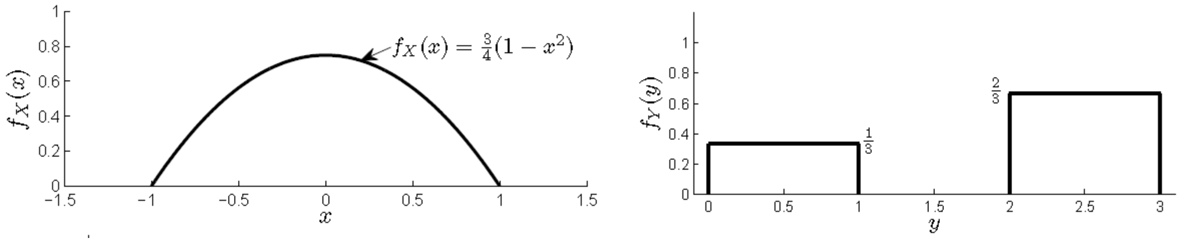
$$
\begin{aligned}
f_W(w) &= 2\lambda^2 \int_0^{\infty} \frac{ue{-}u}{\lambda(1+2w)} \frac{du}{\lambda(1+2w)} \\
&= \frac{2\lambda^2}{\lambda^2(1+2w)^2 \Gamma(2)} \\
&= \frac{2}{(1+2w)^2} \quad ; 1 < w < \infty
\end{aligned}
$$

$$
\begin{aligned}
P(W=1) = P(X < 2Y) &= \int_{y=0}^{\infty} \int_{x=2y}^{\infty} \lambda^2 e^{-\lambda(x+y)} dx dy \\
&= \lambda^2 \int_0^{\infty} e^{-\lambda y} \left. \frac{e^{-\lambda x}}{-\lambda} \right]_{2y}^{\infty} dy \\
&= \lambda \int_0^{\infty} e^{-\lambda y} e^{-2\lambda y} dy \\
&= \lambda \int_0^{\infty} e^{-3\lambda y} dy \\
&= \lambda \left. \frac{e^{-3\lambda y}}{-3\lambda} \right]_0^{\infty} \\
&= \frac{1}{3}
\end{aligned}
$$

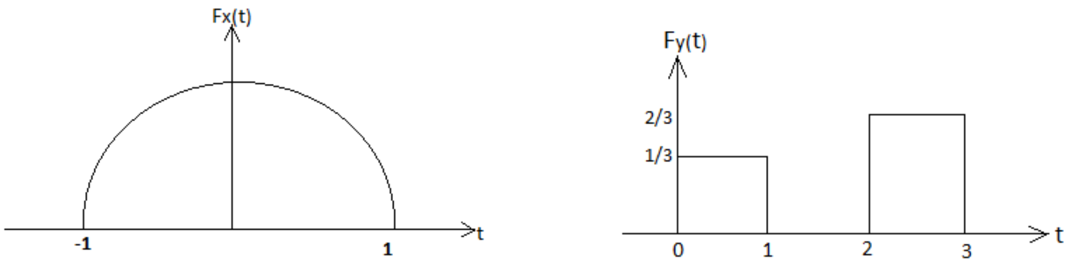


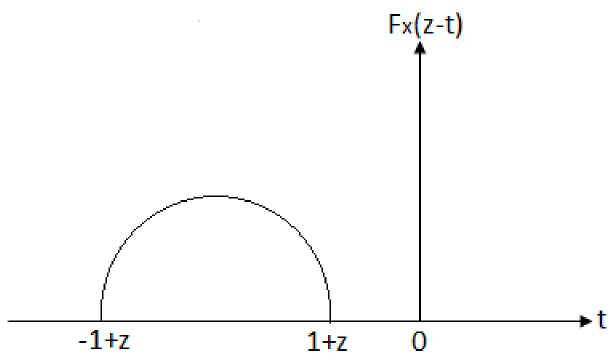


9. Determine $f_Z(z)$ for Z = X+Y , where X and Y are two independent random variables with probability densities as shown in the figure



Solution: Z=X+Y
$f_Z(z) = f_X(x) * f_Y(y)$





case 1:

$$1 + z < 0 \Rightarrow z < -1$$
$$\therefore f_Z(z) < 1 \Rightarrow z < 0$$

case 2:

.
$$\left.\begin{array}{l} 1 + z > 0 \Rightarrow z > -1 \\ 1 + z < 1 \Rightarrow z < 0 \end{array}\right\} -1 < z < 0$$

$$
\begin{aligned}
f_Z(z) &= \int_0^{1+z} \frac{1}{3}.\frac{3}{4}(1-(z-t)^2)dt \\
&= \int_0^{1+z} \frac{1}{4}(1-z^2-t^2+2zt)dt \\
&= \frac{1}{4}(t-z^2t-\frac{t^3}{3}+zt^2)_0^{1+z} \\
&= \frac{1}{4}[(1+z)^2-z^2(1+z)-\frac{(1+z)^3}{3}+z(1+z)^2] \\
&= \frac{1}{4}(2z+z^2-z^3)
\end{aligned}
$$

case 3:

.

$$\left.\begin{array}{l} 1+z>1 \Rightarrow z>0 \\ -1+z<0 \Rightarrow z<1 \end{array}\right\} 0<z<1$$

$$
\begin{aligned}
f_Z(z) &= \int_0^{1} \frac{1}{3}.\frac{3}{4}(1-(z-t)^2)dt = \int_0^{1} \frac{1}{4}(1-z^2-t^2+2zt)dt \\
&= \frac{1}{4}(t-z^2t-\frac{t^3}{3}+zt^2)_0^{1} \\
&= \frac{1}{4}[1-z^2-\frac{(1}{3}+z] \\
&= \frac{1}{4}[\frac{2}{3}-z^2+z]
\end{aligned}
$$

case 4:

.

$$\left.\begin{array}{l} 1+z>2 \Rightarrow z>1 \\ -1+z<1 \Rightarrow z<2 \end{array}\right\} 1<z<2$$

$$
\begin{aligned}
f_Z(z) &= \int_{-1+z}^{1} \frac{1}{3}.\frac{3}{4}(1-(z-t)^2)dt + \int_2^{1+z} \frac{2}{3}.\frac{3}{4}(1-(z-t)^2)dt \\
&= \frac{1}{4}(t-z^2t-\frac{t^3}{3}+zt^2)_{-1+z}^{1} + \frac{1}{2}(t-z^2t-\frac{t^3}{3}+zt^2)_2^{1+z} \\
&= \frac{-z^3}{12}-\frac{7}{4}z^2-\frac{z}{2}+\frac{4}{3}
\end{aligned}
$$

case 5:

.

$$\left.\begin{array}{l} 1+z>3 \Rightarrow z>2 \\ -1+z<2 \Rightarrow z<3 \end{array}\right\} 2<z<3$$

$$
\begin{aligned}
f_Z(z) &= \int_2^{1+z} \frac{2}{3}.\frac{3}{4}(1-(z-t)^2)dt = \frac{1}{2}(t-z^2t-\frac{t^3}{3}+zt^2)_2^{1+z} \\
&= \frac{1}{2}[\frac{-z^3}{3}+2z^2-3z+\frac{4}{3}] \\
&= \frac{-z^3}{6}+z^2-\frac{3}{2}z+\frac{2}{3}
\end{aligned}
$$

case 6:

$$\left.\begin{array}{l} -1+z>2 \Rightarrow z>3 \\ -1+z<3 \Rightarrow z<4 \end{array}\right\} 3<z<4$$

$$f_Z(z) = \int_{-1+z}^{3} \frac{2}{3}.\frac{3}{4}(1-(z-t)^2)dt = \frac{1}{2}(t - z^2t - \frac{t^3}{3} + zt^2)_{-1+z}^{3}$$
$$= (\frac{7}{3}z^3 - 7z^2 + 10z - \frac{16}{3})$$

10. Let X be the lifetime of a certain electric bulb and Y that of its replacement after the failure of the first bulb. Suppose X and Y are independent with common exponential density function with parameter λ. Find the probability that the combined lifetime exceeds 2λ. What is the probability that the replacement outlasts the original component by λ?

Solution:
$X \sim Exponential(\lambda)$,
$Y \sim Exponential(\lambda)$
X and Y are independent
Z=X+Y

$\phi_Z(s) = \phi_X(s).\phi_Y(s) = \left(\frac{\lambda}{\lambda - s}\right)^2$

$f_Z(z) = \lambda^2 z e^{-\lambda z} u(z)$
$\quad = \frac{z}{\lambda^2} e^{-\frac{z}{\lambda}} u(z)$

We need to find $P(z > 2\lambda)$,

$P(z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2} e^{-\frac{z}{\lambda}} dz$

$Put\ x = \frac{z}{\lambda}$
$dx = \frac{dz}{\lambda}$

$P(z > 2\lambda) = \int_{2}^{\infty} xe^{-x} dx$
$\quad = \left.\frac{xe^{-x}}{1}\right]_2^{\infty} - e^{-x}]_2^{\infty}$
$\quad = 3e^{-2}$
$\quad = 0.406$

11. The length of time Z, an airplane, can y is given by Z = αX where X is the amount of fuel in its tank and $\alpha > 0$ is a constant of proportionality. Suppose a plane has two independent fuel tanks so that when one gets empty the other switches on automatically. Because of lax maintenance, a plane takes of with neither of its fuel can checked. Let X_1 be the fuel in the first tank and X_2 the fuel in the second tank. Let X_1 and X_2 be modelled as independent, identically distributed random variables with $f_{X_1}(x) = f_{X_2}(x) = \frac{1}{b}[u(x) - u(x-b)]$. Compute the pdf of Z, the maximum flying time of the plane. If b=100, say in litres, and α = one hour/ ten litres, what is the probability that the plane will y atleast five hours?

Solution: Let $Z_1 = \alpha X_1$ & $Z_2 = \alpha X_2$
$Z = Z_1 + Z_2$
Then,
$f_{Z_1}(z_1) = \frac{1}{\alpha} f_{X_1}(\frac{z_1}{\alpha})$

$$= \frac{1}{\alpha b}\left[u(\frac{z_1}{\alpha}) - u(\frac{z_1}{\alpha} - b)\right]$$

Similarly,

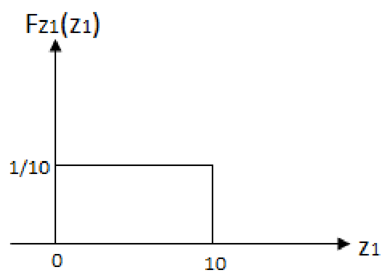
$f_{Z_2}(z_2) = \frac{1}{\alpha b}\left[u(\frac{z_2}{\alpha}) - u(\frac{z_2}{\alpha} - b)\right]$

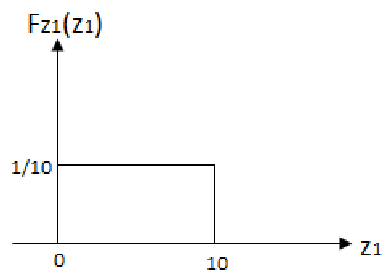
$f_Z(z) = f_{Z_1}(z_1) * f_{Z_2}(z_2)$

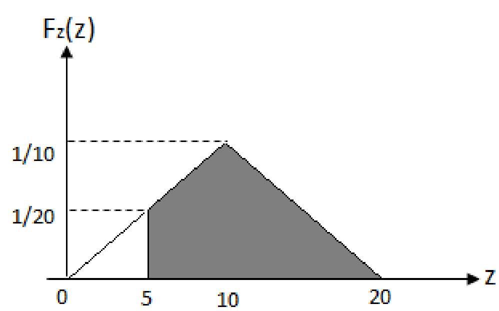
$\because Z_1$ & Z_2 are independent.

$\alpha = \frac{1}{10}$ & $b = 100 \Rightarrow$

$$f_{Z_1}(z_1) = \frac{1}{10}[u(10z_1) - u(10z_1 - 100)]$$
$$f_{Z_2}(z_2) = \frac{1}{10}[u(10z_2) - u(10z_2 - 100)]$$







We need to find $P(Z \geq 5)$ which is indicated by shaded area.

Area of the unshaded triangle $= \frac{1}{2} \times 5 \times \frac{1}{20} = 0.125$

$P(Z \geq 5) = 1 - 0.125$

$= 0.875$