I^{st} Semester M.Tech Microwave & TV Engineering And Signal Processing

Tutorial 7:Joint Moments And Conditional Expectation

1. The random variables X and Y are jointly distributed over the region $0 < x < y < 1$ as

$$f_{XY}(x,y) = \begin{cases} kx & ; 0 < x < y < 1 \\ 0 & ; otherwise \end{cases}$$

(a) Find the correlation coefficient τ_{XY}

(b) Find $E(X|Y = y) and E(Y|X = x)$

Solution:

$$f_{XY}(x,y) = \begin{cases} kx & ; 0 < x < y < 1 \\ 0 & ; otherwise \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dydx = 1$$

$$\int_{x=0}^{1}\int_{y=x}^{1} kxdydx = 1$$

$$k\int_{0}^{1} x(1-x)dx = 1$$

$$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow \frac{k}{6} = 1, \Rightarrow k = 6.$$

$$f_X(x) = \int_x^1 6xdy = 6x(1-x) \; ; 0 < x < 1$$

$$f_Y(y) = 6\int_0^y xdx = 3y^2 \; ; 0 < x < 1$$

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{6x}{3y^2} = \frac{2x}{y^2} \; ; 0 < x < y$$

$$; 0 < y < 1$$

$$f_{Y/X}(y/x) = fXY(x,y)/fX(x) = \frac{6x}{6x(1-x)} = \frac{1}{(1-x)} \; ; 0 < x < 1$$

$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 x.6x(1-x)dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1/2$

$E[Y] = \int_0^1 y.3y^2 dy = \left[\frac{3y^4}{4}\right]_0^1 = \frac{3}{4}$

$E[XY] = \int_{x=0}^1 \int_{y=x}^1 xy.6xdydx = = 6\int_0^1 (\frac{x^2(1-x^2)}{2})dx$

$= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2}{5}$

$E[X^2] = \int_0^1 x^2 6x(1-x)dx = 6\int_0^1 (x^3 - x^4)dx = 6\left[x^4/4 - x^5/5\right]_0^1 = 3/10$

$E[Y^2] = \int_0^1 y^2 3y^2 dy = \frac{3}{5}$

$\sigma_X^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$

$\sigma_Y^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$

(a) $r_{XY} = C_{XY}/\sigma_X\sigma_Y = E[XY] - E[X]E[Y]/\sigma_X\sigma_Y = \frac{2}{5} - \frac{3}{8}/\sqrt{1/20}\sqrt{3/80}$

$= 1/\sqrt{3}$

(b) $E[Y/X = x] = \int_x^1 \frac{ydy}{1-x} = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}\ ; 0 < x < 1$

$E[X/Y = y] = \int_0^y \frac{x2xdx}{y^2} = \frac{2y}{3}\ ; 0 < y < 1$

2. X is a Poisson random variable with parameter $\lambda$and Y is a normal random variable with mean μ and varianceσ^2.x and Y are given to be independent.

   (a).Find the joint characteristic functiom of X and Y.

   (b).DefinedZ=X+Y.Find charecteristics of Z.

Solution:

$X - Poisson(\lambda)$

$\phi_X(\omega_1) = \sum_{k=0}^{\infty} e^{jk\omega}.\frac{e^{-\lambda}\lambda^k}{k_1} = \exp^{-\lambda} \sum_{k=0}^{\infty} \left(\frac{(e^{j\omega}\lambda)^k}{k_1}\right)$

$e^{-\lambda}.e^{\lambda e^{j\omega}} = e^{-\lambda(1-e^{j\omega_1})}$

$\phi_Y(\omega_2) = e^{j\omega_2\mu}.e^{\frac{-\omega_2^2\sigma^2}{2}}$

(a). $\phi_{XY}(\omega_1, \omega_2) = \phi_X(\omega_1)\phi_Y(\omega_2)$, X&Y are independent

$$= e^{-\lambda(1-e^{j\omega_1})}.e^{j\mu\omega_2}.e^{\left(\frac{-\omega_2^2\sigma^2}{2}\right)}$$

(b).$Z = X + Y \implies \phi_Z(\omega) = \phi_X(\omega)\phi_Y(\omega) = e^{-\lambda(1-e^{j\omega})}.e^{j\omega\mu}.e^{\left(\frac{-\omega^2\sigma^2}{2}\right)}$

3. X and Y are independent exponential random variables with common parameter λ.Find

   (a).$E[min(X, Y)]$

   (b).$E[max(X, Y)]$

Solution:

$f_X(x) = \lambda.e^{-x\lambda}u(x)$; $f_Y(y) = \lambda.e^{-\lambda y}u(y)$

$f_{XY}(x, y) = f_X(x) + f_Y(y) = \lambda^2 e^{-\lambda(x+y)}$ $x, y > 0$

(a).$E[min(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} min(x, y)f_{X,Y}dxdy$

$min(X, Y) = X$; $X \leq Y$ $\rightarrow area1$

$= Y$ $X > Y$ $\rightarrow area2$

$E[min(X, Y)] = \int_{x=0}^{\infty}\int_{y=0}^{x} y.\lambda^2 e^{-\lambda(x+y)}dydx + \int_{y=0}^{\infty}\int_{x=0}^{y} x.\lambda^2 e^{-\lambda(x+y)}$

$$= \lambda^2 \int_0^{\infty} e^{-\lambda x}\left(\frac{xe^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda y}}{\lambda^2}\right)dx$$

$$= \int_0^{\infty}(e^{-\lambda x} - e^{-2\lambda x} - \lambda x e^{-2\lambda x})dx$$

$\left[\frac{1}{\lambda} - \frac{1}{2\lambda} + \frac{e^{-2\lambda x}}{4\lambda}\right]_0^{\infty} = \frac{1}{\lambda}(1 - .5 - .25) = \frac{1}{4\lambda}$

$E[min(X, Y)] = \frac{2}{4\lambda} = \frac{1}{2\lambda}$

(b). $E[max(2X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} max(2x,y) f_{XY}(x,y)dxdy$

$= \lambda^2\left[2\int_0^{\infty} xe^{-\lambda x}\int_0^{2x} e^{-\lambda y}dydx \ + \int_0^{\infty} ye^{-\lambda y}\int_0^{y/2} e^{-\lambda x}dxdy\right]$

$= \lambda^2\left[2\int_0^{\infty} xe^{\lambda x}\frac{e^{y-\lambda}}{-\lambda}dx + \int_0^{\infty} ye^{-\lambda y}\frac{e^{\lambda y/2-1}}{-\lambda}dy\right]_0^{y/2}$

$= \frac{\lambda^2}{\lambda}\left[2\int_0^{\infty}(xe^{-\lambda x} - xe^{-3x}dx + \int_0^{\infty}(ye^{-\lambda y} - ye^{-3\lambda y/2})dy\right]$

$= \lambda\left[2\left[\frac{xe^{-\lambda x}}{-\lambda}\right]\right]_0^{\infty} - \int_0^{\infty}\frac{e^{-\lambda x}}{-\lambda}dx - \left[\frac{xe^{-3\lambda x}}{-3\lambda}\right]_0^{\infty} + \int_0^{\infty}\frac{e^{-3\lambda x}}{-3\lambda}dx + \left\{\frac{ye^{-\lambda y}}{-\lambda}\right\}_0^{\infty}$

$- \int_0^{\infty}\frac{e^{-\lambda y}}{-\lambda}dy - \left[\frac{ye^{-3\lambda y/2}}{-3\lambda/2}\right]_0^{\infty} + \int_0^{\infty}\frac{e^{-3\lambda y/2}}{-3\lambda/2}dy$

$= \lambda\left[2\left\{\frac{1}{\lambda^2} - \frac{1}{9\lambda^2}\right\} + \left\{\frac{1}{\lambda^2} - \frac{2}{3\lambda}\frac{2}{3\lambda}\right\}\right] = \lambda\left\{\frac{16}{9\lambda^2} + \frac{5}{9\lambda^2}\right\} = \frac{7}{3\lambda}$

4. Show that, if the random variables X and Y are $N(0,0,\sigma^2,\sigma^2,r)$ then

(a) $E\{f_{\frac{Y}{X}}(y/x)\} = \frac{1}{\sigma\sqrt{2\pi(2-r^2)}} exp\{-\frac{-r^2x^2}{2\sigma^2(2-r^2)}\}$

(b) $E\{f_X(x)f_Y(y)\} = \frac{1}{2\pi\sigma^2\sqrt{4-r^2}}$

Solution:

$X,Y \sim N(0,0,\sigma^2,\sigma^2,r)$

$\Rightarrow f_{XY}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-r^2}}\exp\left\{\frac{-1}{2\sigma^2(1-r^2)}(x^2 - 2rxy + y^2)\right\}$

$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-y^2}{2\sigma^2}}$ $\quad$ $X\&Y$ are individually gaussian

$f_{\frac{Y}{X}}(\frac{y}{x}) = \frac{1}{\sqrt{2\pi\sigma^2(1-r^2)}}\exp\left\{\frac{-(y-rx)^2}{2\sigma^2(1-r^2)}\right\}$

(a) $f_{\frac{Y}{X}}(\frac{y}{x})$ is considered as a function of y, since we find $f_{\frac{Y}{X}}(\frac{y}{x=x_1})$.Each value of x is taken as constant and for that given value of x,we find $f_{\frac{Y}{X}}(\frac{y}{x=x_1})$.if we take a slice from the $3D$ gaussian cone, then area of that curve

$$= \int_{y=-\infty}^{\infty} f_{XY}(-2,y)dy = f_X(-2)$$

So if we take each point in that curve $= f_{XY}(-2,y)$ and divide it with the area of the curve,$f_X(-2)$,we get $f_{\frac{Y}{X}}(\frac{y}{x=-2})$ and the normalized curve corresponds to this $x = -2$

.

Hence, to find $E[f_{\frac{Y}{X}}(\frac{y}{x})]$, consider $f_{\frac{Y}{X}}(\frac{y}{x})$ as $g(y)$ therefore we need $f_Y(y)$ inside the integral.

$$E[f_{\frac{Y}{X}}(\frac{y}{x})] = \int_{-\infty}^{\infty} f_{\frac{Y}{X}}(\frac{y}{x}) f_Y(y) dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(1-r^2)}} e^{\frac{-(y-rx)^2}{2\sigma^2(1-r^2)}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-y^2}{2\sigma^2}} \, dy$$

$$E[f_{\frac{Y}{X}}(\frac{y}{x})] = \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2} \frac{y^2-2rxy+r^2x^2+y^2}{1-r^2}} dy$$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} e^{\frac{\{-(y^2(2-r^2)-2rxy+r^2x^2\}}{2\sigma^2(1-r^2)}} \, dy$$

Do completing the square,

$a = \sqrt{2-r^2}\, y$; $2ab = \sqrt{2-r^2}\, b = rx \Rightarrow b = \frac{rx}{\sqrt{2-r^2}}$

$r^2x^2 = \frac{r^2x^2}{2-r^2} + k \Rightarrow k = r^2x^2(1 - \frac{1}{2-r^2}) = \frac{r^2x^2(1-r^2)}{2-r^2}$

$$exp[\frac{-\{y^2(2-r^2)-2rxy+r^2x^2\}}{2\sigma^2(1-r^2)}] = exp[\frac{-(\sqrt{2-r^2}\, y - \frac{rx}{\sqrt{2-r^2}})^2}{2\sigma^2(1-r^2)} - \frac{r^2x^2}{2\sigma^2}]$$

$$E[f_{\frac{Y}{X}}(\frac{y}{x})] = \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} e^{\frac{-r^2x^2}{2\sigma^2(2-r^2)}} \int_{-\infty}^{\infty} exp\{\frac{-1}{2\sigma^2(1-r^2)}(y\sqrt{2-r^2} - \frac{xr}{\sqrt{2-r^2}})^2\} \, dy$$

Put $u = y\sqrt{2-r^2} - \frac{rx}{\sqrt{2-r^2}} \Rightarrow du = \sqrt{2-r^2}dy$

$$E[f_{\frac{Y}{X}}(\frac{y}{x})] = \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} e^{\frac{-r^2x^2}{2\sigma^2(2-\sigma^2)}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2\sigma^2(1-r^2)}} \frac{du}{\sqrt{2-r^2}}$$

$$= \frac{1}{2\pi\sigma^2\sqrt{2-r^2}} e^{\frac{-r^2x^2}{2\sigma^2(2-\sigma^2)}} \sqrt{2\pi\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2(2-r^2)}} e^{\frac{-r^2x^2}{2\sigma^2(2-r^2)}}$$

$$= \frac{1}{\sigma\sqrt{2\pi(2-r^2)}} exp\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\}$$

(b)$E[f_X(x)f_Y(y)] = \int_{-\infty}^{\infty} f_X(x)f_Y(y)f_{XY}(x,y) \, dx \, dy$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-y^2}{2\sigma^2}} \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} e^{\frac{-1(x^2-2rxy+y^2)}{2\sigma^2(1-r^2)}} \, dx \, dy$$

$$= \frac{1}{(2\pi\sigma^2)^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{\frac{-1}{2\sigma^2}[x^2 + \frac{x^2}{1-r^2} + y^2 + \frac{y^2}{1-r^2} -$$

$\frac{2rxy}{1-r^2}]\}\ dx\ dy$

$= \frac{1}{(2\pi\sigma^2)^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{\frac{-1}{2\sigma^2(1-r^2)}[x^2(2-r^2) +$

$y^2(2-r^2) - 2rxy]\}\ dx\ dy$

Do complete the square

$$a = x\sqrt{2-r^2}\ ;\ 2ab = ry = \sqrt{2-r^2}\, b \Rightarrow b = \frac{ry}{\sqrt{2-r^2}}$$

$$y^2(2-r^2) = \frac{y^2r^2}{2-r^2} + k \Rightarrow k = y^2\{2-r^2-\frac{r^2}{2-r^2}\}$$

$$= y^2\{\frac{4-4r^2+r^4-r^2}{2-r^2}\} = \frac{y^2(r^4-5r^2+4)}{(2-r^2)} = \frac{y^2(r^2-4)(r^2-1)}{2-r^2}$$

$$\frac{-\{x^2(2-r^2)+y^2(2-r^2)-2rxy\}}{2\sigma^2(1-r^2)} = \frac{-(x\sqrt{2-r^2}-\frac{ry}{\sqrt{2-r^2}})^2}{2\sigma^2(1-r^2)} - \frac{y^2(4-r^2)}{2\sigma^2(2-r^2)}$$

$$E[f_X(x)f_Y(y)] = \frac{1}{(2\pi\sigma^2)^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} e^{\frac{-y^2(4-r^2)}{2\sigma^2(2-r^2)}} \int_{-\infty}^{\infty} e^{\frac{-\{x\sqrt{2-r^2}-\frac{ry}{\sqrt{2-r^2}}\}^2}{2\sigma^2(1-r^2)}}\ dx\ dy$$

Put $u = x\sqrt{2-r^2} - \frac{yr}{\sqrt{2-r^2}} \Rightarrow du = dx\sqrt{2-r^2}$

$$E[f_X(x)f_Y(y)] = \frac{1}{(2\pi\sigma^2)^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} e^{\frac{-y^2(4-r^2)}{2\sigma^2(2-r^2)}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2\sigma^2(1-r^2)}} \frac{du}{\sqrt{2-r^2}}\ dy$$

$$= \frac{1}{(2\pi\sigma^2)^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} \frac{e^{\frac{-y^2(4-r^2)}{2\sigma^2(2-r^2)}}}{\sqrt{2-r^2}} \sqrt{2\pi\sigma^2(1-r^2)}\ dy$$

$$= \frac{\sqrt{2\pi\sigma^2(1-r^2)}}{(2\pi\sigma^2)^2\sqrt{1-r^2}\sqrt{2-r^2}} \int_{-\infty}^{\infty} exp\{\frac{-y^2(4-r^2)}{2\sigma^2(2-r^2)}\}\ dy$$

Put $u = y\sqrt{4-r^2} \Rightarrow du = dy\ \sqrt{4-r^2}$

$$E[f_X(x)f_Y(y)] = \frac{\sqrt{2\pi\sigma^2}}{(2\pi\sigma^2)^2\ \sqrt{2-r^2}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2\sigma^2(2-r^2)}} \frac{du}{\sqrt{4-r^2}}$$

$$= \frac{\sqrt{2\pi\sigma^2}}{(2\pi\sigma^2)^2\ \sqrt{4-r^2}} \sqrt{2\pi\sigma^2}$$

$$= \frac{1}{2\pi\sigma^2\sqrt{4-r^2}}$$

5. Show that

$$E\{Y/X \le 0\} = \frac{1}{F_X(0)} \int_{-\infty}^{0} E\{Y/X\} f_X(x) dx$$

Solution:

$$f_{XY}(x,y/x\le 0) \;=\; \frac{f_{XY}(x,y)\ over\ x\le 0}{P(x\le 0)}$$

$$E[Y/X\le 0] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\frac{yf_{XY}(x,y)\ overx\le 0}{P(x\le 0)}\,dy\,dx$$

$$= \frac{1}{F_X(0)}\int_{-\infty}^{0}\int_{-\infty}^{\infty} y\ f_X(x)\ f_{Y/X}(y/x)\ dy\ dx$$

$$= \frac{1}{F_X(0)}\int_{-\infty}^{0} f_X(x)\int_{-\infty}^{\infty} y\ f_{Y/X}(y/x)\ dy\ dx$$

$$= \frac{1}{F_X(0)}\int_{-\infty}^{0} f_X(x)\ E[Y/X]\ dx$$

6. Let X be a random variable with Poisson distribution.Find conditional expectation of X given that X is an even number.

Solution:

$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$

$P(X=k\mid\ X\ is\ even)=\frac{2e^{-\lambda}\lambda^k}{k!(1+e^{-2\lambda})}=\frac{\lambda^k}{k!(\frac{e^{\lambda}+e^{-\lambda}}{2})}=\frac{\lambda^k}{k!cosh\lambda}$

$$P(X=k\mid\ X\ is\ even)=\sum_{i=2k} i\frac{\lambda^i}{i!cosh\lambda}=\sum_0^{\infty}2k\frac{\lambda^{2k}}{(2k)!cosh\lambda}$$

$$=\frac{1}{cosh\lambda}\sum_{k=0}^{\infty}\frac{2k.\lambda^{2k}}{(2k)!}=\frac{1}{cosh\lambda}\sum_{k=1}^{\infty}\frac{2k.\lambda^{2k}}{(2k)!}\qquad since\ first\ term\ is\ 0$$

$$=\frac{1}{cosh\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{2k}}{(2k-1)!}=\frac{\lambda}{cosh\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{2k-1}}{(2k-1)!}$$

$$=\frac{\lambda}{cosh\lambda}.sinh\lambda=\lambda.tanh\lambda\qquad since\ \frac{\lambda}{1!}+\frac{\lambda^3}{3!}+...=sinh\lambda$$

7. Compute the joint characteristic function of the discrete function of the discrete random variables X and Y if the joint PMF is

$$P_{X,Y}(k,l)=\begin{cases}\frac{1}{3} & ;k=l=0\\ \frac{1}{6} & ;k=\pm,l=0\\ \frac{1}{6} & ;k=l=\pm 1\\ 0 & ;otherwise\end{cases}$$

Also find the correlation coefficient between X and Y using the characteristic function.

Solution:

$\emptyset_{XY}(\omega_1,\omega_2) = \sum_{i=-1}^{1}\sum_{j=-1}^{1} e^{j(\omega_1 x_i+\omega_2 y_j).P_{XY}(x_i,y_j)}$

$= e^{j(-\omega_1-\omega_2)}.\frac{1}{6} + e^{j(-\omega_1)}.\frac{1}{6} + e^{j0}.\frac{1}{3} + e^{j(\omega_1)}.\frac{1}{6} + e^{j(\omega_1+\omega_2)}.\frac{1}{6}$

$= \frac{1}{6}[e^{j\omega_1} + e^{j-\omega_1} + e^{j(\omega_1+\omega_2)} + e^{-j(\omega_1+\omega_2)}] + \frac{1}{3}$

$= \frac{1}{3}[1 + cos\omega_1 + cos(\omega_1+\omega_2)]$

$E[X] = (-j)\frac{\delta}{\delta\omega_1}\emptyset_{XY}(\omega_1,\omega_2)\mid_{\omega_1=\omega_2=0} = (-j)[\frac{1}{3}\{-sin\omega_1 - sin(\omega_1+\omega_2)\}]\mid_{\omega_1=\omega_2=0}$

$= 0$

$E[Y] = (-j)\frac{\delta}{\delta\omega_1}\emptyset_{XY}(\omega_1,\omega_2)\mid_{\omega_1=\omega_2=0} = (-j)[\frac{1}{3}\{-sin(\omega_1+\omega_2)\}]\mid_{\omega_1=\omega_2=0}$

$= 0$

$E[X^2] = (-j)^2\frac{\delta^2}{\delta\omega_1^2}\emptyset_{XY}(\omega_1,\omega_2)\mid_{\omega_1=\omega_2=0} = (-1)\frac{\delta}{\delta\omega_1}[\frac{1}{3}\{-sin\omega_1 - sin(\omega_1+\omega_2)\}]\mid_{\omega_1=\omega_2=0}$

$= \frac{-1}{3}[-cos\omega_1 - cos(\omega_1+\omega_2)]\mid_{\omega_1=\omega_2=0} = \frac{-1}{3}(-1-1) = \frac{2}{3}$

$E[Y^2] = (-j)^2\frac{\delta^2}{\delta\omega_2^2}\emptyset_{XY}(\omega_1,\omega_2)\mid_{\omega_1=\omega_2=0} = (-1)\frac{\delta}{\delta\omega_2}[\frac{1}{3}\{-sin(\omega_1+\omega_2)\}]\mid_{\omega_1=\omega_2=0}$

$= \frac{-1}{3}[-cos(\omega_1+\omega_2)]\mid_{\omega_1=\omega_2=0} = \frac{-1}{3}(-1) = \frac{1}{3}$

$\sigma_Y^2 = E[Y^2] \qquad since\ \ E[Y] = 0$
$= \frac{1}{3}$

$E[XY] = (-j)^2\frac{\delta^2}{\delta\omega_1\delta\omega_2}\emptyset_{XY}(\omega_1,\omega_2)\mid_{\omega_1=\omega_2=0} = (-1)\frac{\delta}{\delta\omega_2}[\frac{1}{3}\{-sin\omega_1 - sin(\omega_1+\omega_2)\}]\mid_{\omega_1=\omega_2=0}$

$= -1\times\frac{1}{3}\times -cos(\omega_1+\omega_2)\mid_{\omega_1=\omega_2=0} = \frac{1}{3}$
$C_{XY} = E[XY] \qquad since\ \ E[X] = 0E[Y] = 0$
$= \frac{1}{3}$
$r_{XY} = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{\frac{1}{3}}{\sqrt{\frac{1}{3}\times\frac{2}{3}}} = \frac{1}{\sqrt{2}}$

8. Let X and Y be jointly continuous random variables with joint density

$$f_{XY}(x,y) = \begin{cases} e^{-x-y} & ; x,y \geq 0 \\ 0 & ; otherwise \end{cases}$$

Compute the conditional expectation of $X+Y$ given that $X<Y$.

Solution:

$$f_{XY}(x,y) = \begin{cases} e^{-x-y} & ; x,y \geq 0 \\ 0 & ; otherwise \end{cases}$$

$E[X+Y/X<Y] = \frac{E[X+Y, X<Y]}{P(X<Y)}$

$E[X+Y, X<Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f_{XY}(x,y)dxdy \qquad over x<y$

$= \int_{x=0}^{\infty}\int_{y=x}^{\infty}(x+y)e^{-(x+y)}dxdy \;=\; \int_{x=0}^{\infty}[\int_{y=x}^{\infty}(x+y)e^{-(x+y)}dy]dx$

$= \int_{x=0}^{\infty}\int_{u=2x}^{\infty}ue^{-u}dudx \;=\; \int_0^{\infty}\left[\frac{ue^{-u}}{-1}\right]_{2x}^{\infty} \;-\; \int_{2x}^{\infty}\frac{e^{-u}}{-1}du \;\; dx$

$= \int_0^{\infty}[2x\,e^{-2x} \;-\; e^{-u}]_{2x}^{\infty}\,dx \;=\; 2\int_0^{\infty}xe^{-2x}dx \;+\; \int_0^{\infty}e^{-2x}dx$

$= 2(\frac{1}{4}) + \left[\frac{e^{-2x}}{-2}\right]_0^{\infty} \;=\; \frac{1}{2}+\frac{1}{2} \;=\; 1$

$E[X+Y/X<Y] = \frac{1}{P(X<Y)} \;=\; \frac{1}{1/2} \;=\; 2$

9. Certain sums of iid random random variables are familier random variables themselves. When $W = X_1 + ... + X_n$ is a sum of n iid random variables, show that:

   (a) If X_i is Bernoulli (p), W is binomial (n,p).

   (b) If X_i is geometric (p), W is Pascal (n,p).

   (c) If X_i is exponential (λ), W is Erlang (n,λ).

Solution:

(a) Bernolli $\rightarrow P(X_i = k) \;=\; p^k q^{1-k}\;; k = 0,1$

$\Phi_{X_i}(\omega) = \;\Sigma_{k=0}^{1}\, e^{j\omega k}P(X_i = k) = \;\Sigma_{k=0}^{1}\, e^{j\omega k}(p^k q^{1-k})$

$= q + pe^{j\omega}$

$W = \Sigma_{i=1}^{n}X_i \Rightarrow\; \Phi_W(\omega) = \Pi_{i=1}^{n}\Phi_{Xi}(\omega) = [\Phi_{Xi}(\omega]^n$

$=(pe^{j\omega}+q)^n$

Suppose z is binomial(n,p).Then $P(z=k) = \binom{n}{k}p^k q^{n-k}$

$\Phi_z(\omega) = \Sigma_{k=0}^{n}\binom{n}{k}p^k q^{n-k}e^{j\omega k} = \Sigma_{k=0}^{n}\binom{n}{k}(pe^{j\omega})^k q^{n-k}$

$=(pe^{j\omega}+q)^n$

$\Phi_W(\omega) = \Phi_z(w) \;\Rightarrow\; W$ is binomial(n,p)

(b) X_i is geometric $\Rightarrow$ $P(X_i = k)pq^{k-1}$;$k = 1, 2, ...\infty$;$p + q = 1$

$\Phi_{X_i}(\omega) = \Sigma_{k=1}^{\infty} e^{j\omega k}pq^{k-1} = pe^{j\omega} + pe^{2j\omega}q + pe^{3j\omega}q^2 + ...$

It forms a GP with $a = pe^{j\omega}$, $r = qe^{j\omega}$ which converges to $\frac{a}{1-r}$

$\Phi_{X_i}(\omega) = \frac{pe^{j\omega}}{1-qe^{j\omega}} = \frac{p}{e^{j\omega}-q}$

$\Phi_W(\omega) = [\Phi_{X}{}_i(\omega)]^n = (\frac{p}{e^{-j\omega}-q})^n$

Suppose Z is Pascal $(n, p) \Rightarrow P(Z = k) = \binom{k-1}{n-1}p^nq^{k-n}$; $k = n, n + 1, ..$;
$p + q = 1$

$\Phi_Z(\omega) = \Sigma_{k=n}^{\infty}\binom{k-1}{n-1}p^nq^{k-n}e^{j\omega k}$, Put $u = k - n$; $k = u + n$

$= \Sigma_{u=0}^{\infty}\binom{u+n-1}{u}p^nq^ue^{j\omega(u+n)}$

$= \Sigma_{u=0}^{\infty}\binom{u+n-1}{u}(pe^{j\omega})^n(qe^{j\omega})^u \longrightarrow$ expansion of the form$(a - b)^{-n}$

$(1 - q)^{-r} = \Sigma_{n=0}^{\infty} \binom{-r}{n}(-q)^n$

$= \Sigma_{k=r}^{\infty}\binom{k-1}{r-1}q^{k-r}$

$\Phi_Z(\omega) = \Sigma_{k=n}^{\infty}\binom{k-1}{n-1}p^nq^{k-n}e^{j\omega(k-n)}e^{j\omega n}$

$= (pe^{j\omega})^n\Sigma_{k=n}^{\infty}\binom{k-1}{n-1}(qe^{j\omega})^{k-n} = (pe^{j\omega})^n(1 - qe^{j\omega})^{-n}$

$\Phi_W(\omega) = (\frac{pe^{j\omega}}{1-qe^{j\omega}})^n = (\frac{p}{e^{-j\omega}-q})^n$

$\Phi_W(m) = \Phi_Z(\omega) \Rightarrow$ W is Pascal (n, p)

(c) $X_i \rightarrow exponential(\lambda) \rightarrow f_{X_i}(x_i) = \lambda e^{-\lambda x}u(x)$

$$\phi_{X_i}(\omega) = \int_{-\infty}^{\infty} e^{j\omega X_i} f_{X_i}(x_i)dx = \int_0^{\infty} e^{j\omega x}.\lambda e^{-\lambda x}dx$$

$$= \lambda. \int_0^{\infty} e^{(j\omega-\lambda)x}dx = \lambda. \left[\frac{e^{-(\lambda-j\omega)x}}{-(\lambda-j\omega)}\right]_0^{\infty} = \frac{\lambda}{\lambda-j\omega}$$

$$\phi_W(\omega) = \left(\frac{\lambda}{\lambda-j\omega}\right)^n$$

Suppose Z is Erlang (n, λ)

$$f_Z(z) = \frac{z^{n-1}.\lambda^n.e^{-\lambda z}}{(n-1)!}u(z)$$

$$\phi_Z(\omega) = \int_0^{\infty} e^{j\omega z}\frac{z^{n-1}.\lambda^n.e^{-\lambda z}}{(n-1)!}dz = \frac{\lambda^n}{(n-1)!}\int_0^{\infty} z^{n-1}e^{-(\lambda-j\omega)z}dz$$

$$= \frac{\lambda^n}{(n-1)!}\frac{1}{(\lambda-j\omega)^n}\int_0^{\infty} u^{n-1}e^{-u}du = \frac{\lambda^n}{(\lambda-j\omega)^n} = \left(\frac{\lambda}{\lambda-j\omega}\right)^n$$

$\phi_W(\omega) = \phi_Z(\omega) \rightarrow W$ is Erlong (n, λ)

10. Let

$$Y = \frac{1}{N}\sum_{i=1}^{N} X_i$$

where theX_i are independent, identically distributed Cauchy random variables with

$fX_i(x) = \frac{1}{\pi[1+(x-\mu)^2]}$ $\quad\quad i = 1, 2, \cdots, N$

Show that the pdf of Y is

$$f_Y(x) = \frac{1}{\pi[1+(x-\mu)^2]}$$

that is, identical to the pdf of X_i's and independent of N

Solution:

$Y = \frac{1}{N}\sum_{i=1}^{N} X_i$

$$f_{X_i}(x) = \frac{1}{\pi[1+(x-\mu)^2]} \quad ; \quad i = 1, 2, \ldots, N$$

$$= \frac{1}{2\pi}\left[\frac{2}{1+(x-\mu)^2}\right]$$

$$e^{-a|x|} \rightleftharpoons \frac{2a}{a^2+\omega}$$

$$\Rightarrow \frac{2a}{a^2+\omega^2} \rightleftharpoons 2\pi e^{-a|x|}$$

$$\frac{2a}{a^2+(x-x_0)^2} \rightleftharpoons 2\pi e^{-a|\omega|} e^{-j\omega x_0}$$

Since characteristics function is FT with $e^{j\omega x}$ instead of $e^{-j\omega x}\frac{2a}{a^2+(x-x_0)^2}$ gives a characteristic function $2\pi e^{-a|\omega|}e^{j\omega x_0}$.

$$\therefore\ \phi_{X_i}(\omega) = e^{-|\omega|}\ e^{j\omega\mu}$$

$$\phi_Y(\omega) = E[e^{j\omega y}]$$

$$= E[e^{j\omega\frac{1}{N}\sum_{i=1}^{N} X_i}]$$

$$= E[e^{j\omega\frac{X_1}{N}}]\ E[e^{j\omega\frac{X_2}{N}}]\ E[e^{j\omega\frac{X_3}{N}}]\ \cdots\ E[e^{j\omega\frac{X_N}{N}}$$

$(\because X_{i's}$ are independent).

$$= \pi_{i=1}^{N} \ \phi_{X_i}\left(\frac{W}{N}\right)$$

$$= \left[\phi_{X_i}\left(\frac{W}{N}\right)\right]^N$$

$$= \left[e^{j\frac{W}{N}\mu} \ e^{-\left|\frac{W}{N}\right|}\right]^N$$

$$= e^{j\omega\mu} \ e^{-|\omega|}$$

$$= \phi_{X_i}(\omega)$$

$\therefore 33 f_Y(y) = f_{X_i}(x)$

11. In a game played on a 3 square board,a coin is tossed repeatedly. Heads allows the player to move to the next square and tails to jump over the next square. if the end of the board is reached the reward is \$10.However,there is a trap on one of the squares, placed randomly with uniform probability.If the player falls into the trap the game stops and he has to pay a penalty of \$10.Let X be the gain or loss made by the player and let Y be the number of the square containing the trap.Compute $E(X/Y)$and$E(X)$

Solution:

$Y \longrightarrow$ number of the square with the trap

When $Y = 1$,

$W_{in} = T +$ any combo

$P(Win) = P(T) = \frac{1}{2}$

$P(Loss) = 1 - P(Win) = \frac{1}{2}$

$E[X/Y_{=1}] = 10P(Win) + (-10)P(Loss) = 10 \times \frac{1}{2} - 10 \times \frac{1}{2} = 0$

When $Y = 2$,

Loss= $\{HH, TT\}$

P(Loss)= $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} = \frac{3}{4}$

P(Win)= $\frac{1}{4}$

$E[X/Y_{=2}] = 10 \times \frac{1}{4} + (-10) \times \frac{3}{4} = -5$

When Y= 3,

$W_{in} = \{HHT, TT\}$

$P(W_{in}) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$P(Loss) = \frac{5}{8}$

$E[X/Y_{=3}] = 10 \times \frac{3}{8} + (-10) \times \frac{5}{8} = -2.5$

$E[X] = P(Y = 1)E[X/Y_{=1}] + P(Y = 2)E[X/Y_{=2}] + P(Y = 3)E[X/Y_{=3}]$

$= \frac{1}{3}(0 - 5 - 2.5) = -2.5$

12. A bag contains 3 coins,but one has heads on both sides.Two coins are drawn and tossed.Find $E(X|Y)$ if X is the number of heads and Y is the number of genuine coins among the drawn ones.

Solution:

RV Y can take two values→ 1 and 2

$$\begin{aligned} P(Y = 1) &= \frac{No.\ of\ ways\ to\ select\ 1\ genuine\ coin \times No.\ of\ ways\ for\ 1\ bad\ coin}{3C_2} \\ &= \frac{2C_1 \times 1C_1}{3C_2} \\ &= \frac{2}{3} \\ P(Y = 2) &= \frac{No.\ of\ ways\ to\ select\ 2\ genuine\ coins}{No.\ of\ ways\ to\ select\ 2\ coin} \\ &= \frac{2C_2}{3C_2} \\ &= \frac{1}{3} \end{aligned}$$

When Y=1,X can take values 1 or 2

P(X=1)=P(X=2)=$\frac{1}{2}$

Since only possibilities are $\{HH, HT\}$

E$(X|Y) = \frac{1}{2} \times 1 + \frac{1}{2} \times 2 = \frac{3}{2}$

When Y=2, X can take values 0,1 or 2

P$(X=0)=P(X=2)=\frac{1}{4}$

P$(X=1)=\frac{1}{2}$

Since Possibilities are $\{HH, HT, TH, TT\}$

E$(X|Y)=2=\frac{1}{4}\times 0+\frac{1}{2}\times 1+\frac{1}{4}\times 2=1$

13. Let X and Y be independent random variables having Poisson distribution with parametres λ_1 and λ_2 respectively.
    Compute $E(X|(X+Y)=k)$

Solution:

$$E(X|(X+Y)=k) = \sum_{x=0}^{k} x.P(X=x|(X+Y)=k)$$

P$(X=x|X+Y=k)=\frac{P(X=x,(X+Y)=k)}{P(X+Y)=k}=\frac{P(X=x)P(Y=k-x)}{P(X+Y=k)}$, as X and Y are independent.

$$\begin{aligned} P(X=x|(X+Y)=k) &= \frac{\frac{e^{-\lambda_1}{\lambda_1}^x}{x!}\cdot\frac{e^{-\lambda_2}{\lambda_2}^{k-x}}{(k-x)!}}{e^{-(\lambda_1+\lambda_2)}\,(\lambda_1+\lambda_2)^k} \\ &= kC_x(\lambda_1)^x(\lambda_2)^{k-x}(\lambda_1+\lambda_2)^{-k} \end{aligned}$$

$$\begin{aligned} E(X|(X+Y)=k) &= \sum_{x=0}^{k} x.\ kC_x(\lambda_1)^x(\lambda_2)^{k-x}(\lambda_1+\lambda_2)^{-k} \\ &= \sum_{x=1}^{k}\frac{k!}{(k-x)!(x-1)!}(\lambda_1)^x(\lambda_2)^{k-x}(\lambda_1+\lambda_2)^{-k} \\ &= (\lambda_1+\lambda_2)^{-k}\ .\ k\ \lambda_1\sum_{x=1}^{k}\frac{(k-1)!}{(k-x)!(x-1)!}(\lambda_1)^{x-1}(\lambda_2)^{k-x} \end{aligned}$$

Put $(x-1)=u$

$$\Rightarrow \sum_{u=0}^{k-1} \frac{(k-1)!(\lambda_1)^u \ (\lambda_2)^{(k-1-u)}}{(k-1-u)!u!}$$
$$= (\lambda_1+\lambda_2)^{k-1}$$

So ,

$$\begin{aligned} E(X|(X+Y)=k) &= (\lambda_1+\lambda_2)^{-k}.k\lambda_1.(\lambda_1+\lambda_2)^{k-1} \\ &= k\lambda_1(\lambda_1+\lambda_2)^{-1} \\ &= \frac{k\lambda_1}{\lambda_1+\lambda_2} \end{aligned}$$

$$E(X|(X+Y)=k) = \sum_{x=0}^{k} x.P(X=x|(X+Y)=k)$$

P$(X=x|X+Y=k) = \frac{P(X=x,(X+Y)=k)}{P(X+Y)=k} = \frac{P(X=x)P(Y=k-x)}{P(X+Y=k)}$, as X and Y are independent.