1^{st} Semester M-Tech Microwave & TV Engineering And Signal Processing

Tutorial 9: Random Vectors

1. The random variable Y_1, Y_2, Y_3, Y_4 have the joint pdf

$$f_{Y_1,Y_2,Y_3,Y_4} = \begin{cases} 4 & 0 \le y_1 \le y_2 \le 1, 0 \le y_3 \le y_4 \le 1; \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the marginal pdf of $f_{Y_1,Y_4}(y_1, y_4)$, $f_{Y_2,Y_3}(y_2, y_3)$, $f_{Y_3}(y_3)$.

   (b) Are Y_1, Y_2, Y_3, Y_4 independent random variables ?

Soln:

(a) $f_{Y_1,Y_4}(y_1, y_4) = \int_{y_2=-\infty}^{\infty} \int_{y_3=-\infty}^{\infty} 4dy_3dy_2 = 4\int_{y_1}^{1} \int_{0}^{y_4} dy_3dy_2$
$= 4\int_{y_1}^{1} y_4dy_2 = 4y_4(1 - y_1) \qquad 0 \le y_4 \le 1, 0 \le y_1 \le 1;$

$f_{Y_2,Y_3}(y_2, y_3) = 4\int_{y_1=0}^{y_2} \int_{y_4=y_3}^{1} dy_4dy_1$
$= 4\int_{y_1=0}^{y_2} (1 - y_3)dy_1 = 4(1 - y_3)y_2 \qquad 0 \le y_2 \le 1, 0 \le y_3 \le 1;$

$f_{Y_3}(y_3) = \int_{y_2=0}^{1} 4y_2(1 - y_3)dy_2 = 2(1 - y_3) \qquad 0 \le y_3 \le 1;$

(b) $f_{Y_2}(y_2) = \int_{y_3=0}^{1} 4y_2(1 - y_3)dy_3 = 4y_2(1 - \frac{1}{2}) = 2y_2 \qquad 0 \le y_2 \le 1;$

$f_{Y_1}(y_1) = \int_{y_4=0}^{1} 4y_4(1 - y_1)dy_4 = 2(1 - y_1) \qquad 0 \le y_1 \le 1;$

$f_{Y_4}(y_4) = \int_{y_1=0}^{1} 4y_4(1 - y_1)dy_1 = 2y_4 \qquad 0 \le y_1 \le 1;$

$f_{Y_1}(y_1)f_{Y_2}(y_2)f_{Y_3}(y_3)f_{Y_4}(y_4) = 16y_2y_4(1-y_1)(1-y_3) \ne f_{Y_1,Y_2,Y_3,Y_4}(y_1, y_2, y_3, y_4)$

Y_1, Y_2, Y_3, Y_4 **are not independent**

2. Let a vector $\mathbf{X}$ have E[$\mathbf{X}$] $= 0$ with covariance $\mathbf{K_{XX}}$ given by

$$\mathbf{K_{XX}} = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{bmatrix}$$

Find a linear transformation $\mathbf{C}$ such that $\mathbf{Y} = \mathbf{CX}$ will have

$$\mathbf{K_{YY}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Is $\mathbf{C}$ a unitary transformation ?

Soln:

$$K_{XX} = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{bmatrix}$$

$$E[X] = 0 \qquad\qquad Y = CX$$

$$K_{YY} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow C = ZU^T = \lambda^{\frac{-1}{2}} U^T$$

$$|C - \lambda I| = \begin{vmatrix} 3-\lambda & \sqrt{2} \\ \sqrt{2} & 4-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 5)(\lambda - 2) = 0 \qquad \Rightarrow \qquad \lambda_1 = 5, \lambda_2 = 2$$

$$|C_X - 5I| = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \sim \begin{bmatrix} -\sqrt{2} & 1 \\ \sqrt{2} & -1 \end{bmatrix} \sim \begin{bmatrix} -\sqrt{2} & 1 \\ 0 & 0 \end{bmatrix}$$

$$-\sqrt{2}x_1 + x_2 = 0 \Rightarrow x_2 = \sqrt{2}x_1 \Rightarrow x_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \Rightarrow x_2 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

$$norm = \sqrt{\tfrac{3}{2}}$$

$$|C_X - 2I| = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{2} \\ 10 & \sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$x_1 + \sqrt{2}x_2 = 0 \qquad \Rightarrow x_1 = -\sqrt{2}x_2 \Rightarrow x_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$

$norm = \sqrt{2+1} = \sqrt{3}$

$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$C = \lambda \overset{-1}{\overline{\ 2\ }}\, U^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{15}} & \sqrt{\frac{2}{15}} \\ -\sqrt{\frac{1}{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

C$\neq C^T \Rightarrow$ C is not unitary

3. The 2 X 2 matrix

beginenumerate

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is called a rotation matrix because **y**=**Qx** is the rotation of **x** by the angle θ .Suppose $x = [X_1, X_2]$' is a Gaussian $(0, C_X)$ is a vector where

$$C_X = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}$$

and$\sigma_2^2 \geq \sigma_1^2$. Let **Y=QX**

(a) Find the covariance of Y_1 and Y_2.Show that Y_1 and Y_2 are independent for all θ if $\sigma_2^2 = \sigma_1^2$.

(b) Suppose $\sigma_2^2 = \sigma_1^2$. For what values of θ are Y_1 and Y_2 independent ?

Soln

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$y = Qx; \qquad C_X = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}; \qquad \sigma_2^2 \geq \sigma_1^2$$

(a) $C_Y = QC_XQ^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \sigma_1^2\cos\theta & -\sigma_2^2\sin\theta \\ \sigma_1^2\sin\theta & \sigma_2^2\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2\cos^2\theta + \sigma_2^2\sin^2\theta & \sigma_1^2\cos\theta\sin\theta - \sigma_2^2\cos\theta\sin\theta \\ \sigma_1^2\cos\theta\sin\theta - \sigma_2^2\cos\theta\sin\theta & \sigma_1^2\sin^2\theta + \sigma_2^2\cos^2\theta \end{bmatrix}$$

If $\sigma_1^2 = \sigma_2^2 = \sigma^2, \qquad C_X = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix};$

$\Rightarrow Y_1 \& Y_2$ are uncorrelated

$\Rightarrow Y_1 \& Y_2$ are independent since they are Gaussian RVs

(b) $\sigma_1^2\cos\theta\sin\theta = \sigma_2^2\cos\theta\sin\theta$

$(\sigma_2^2 - \sigma_1^2)\cos\theta\sin\theta = 0$

$\frac{(\sigma_2^2-\sigma_1^2)}{\sin 2\theta} = 0 \qquad (\sigma_2^2 - \sigma_1^2)$ not zero since$\sigma_2^2 \geq \sigma_1^2$

$\sin 2\theta = 0 \qquad \Rightarrow 2\theta = n\pi \qquad \Rightarrow \theta = \frac{n\pi}{2}; n = -\infty ... -1, 0, 1 ...$

4. Show the following one-sided version of Chebyshev inequality: $P(X - \mu \geq a) \leq \frac{\sigma^2}{\sigma^2+a^2}$

Ans : With $a > 0$ and $c \geq 0$

$$
\begin{aligned}
P(X-\mu \geq a) = P(X-\mu+c \geq a+c) \quad &\leq P((X-\mu+c)^2 \geq (a+c)^2) \\
&\leq \frac{E((X-\mu+c)^2)}{(a+c)^2} \\
&= \frac{\sigma^2+c^2}{(a+c)^2}
\end{aligned}
$$

where the first inequality follows from the fact that $a+c > 0$ and the second inequality follows from the markov inequality.
To tighten the bound, we treat $\frac{\sigma^2+c^2}{(a+c)^2}$ as a function of c and find c such that the derivative is 0.The minimum occurs at $c = \frac{\sigma^2}{a}$.Therefore,

$$
\begin{aligned}
P(X-\mu > a) &\leq \frac{\sigma^2 + (\frac{\sigma^4}{a^2})}{(a+\frac{\sigma^2}{a})^2} \\
&= \frac{\sigma^2}{\sigma^2+a^2}
\end{aligned}
$$

5. Let $X_n, n \geq 1$ be a sequence of iid random variables distributed uniformly in [0,1].Consider a random variable
   $Z_n = n\ min(X_1, X_2, ...X_n)$
   Find $\lim_{n\to\infty} P(Z_n \geq z)$,for all z and identify the limiting random variable with this law.
   Ans: For large n,

$$
\begin{aligned}
P(Z_n \geq z) &= \prod_{i=1}^{n} P(X_i \geq \frac{z}{n}) \\
&= (1 - \frac{z}{n})^n \\
&\approx (e^{\frac{-z}{n}}) \\
&= e^{-z}
\end{aligned}
$$

6. A company manufactures toaster ovens. Let the probability that a toaster oven has a dent or scratch be $p = 0 : 05$. Assume different ovens get dentedor scratched independently. In one week the company makes 2000 of these ovens . What is the approximate probability that in this week more than 109 ovens are dented or scratched?
   Ans:We use the central limit theorem.

   Let$X_i$where $(i = 1.....2000)$ denote the state of the i^{th} toaster.

$$
X_i = \begin{cases} 1 & \text{if the toaster dented with probability p} \\ 0 & \text{if the toaster OK with probability } q = 1 - p \end{cases}
$$

$$
\begin{aligned}
P[X_i - 1] &= p = 0.05 \\
P[X_i = 0] &= 1 - p = 0.95 = q \\
Let W &= \sum_{i=1}^{2000} X_i \\
\mu_x i &= 0.05 = p \\
\sigma_x i^2 &= p - p^2 = pq \\
n &= 2000 \\
Then \mu_w &= np = 100 \\
\sigma_w^2 &= npq = 95 \\
f_w(w) &= \frac{1}{\sqrt{(2\Pi \times 95)}} \times e^{\frac{-(w-100)^2}{2\times 95}} \\
P(w \geq 110) &= 1 - F_w(110) \\
&= 1 - G\frac{(110 - 100)}{(\sqrt{9}5)} \\
&= 1 - G(1.02) \qquad = 0.5 - erf(1.02) \\
&= 0.5 - 0.34134 \\
&= 0.15
\end{aligned}
$$

7. Compute the chernoff bound on $P(X \geq a)$where X is a random variable that satisfies the poisson distribution with parameter λ . Ans: $P(X = k) = \frac{(e^{-}\lambda)(\lambda^k)}{k!}$
$k = 0, 1, 2....$

$$
\begin{aligned}
P(X > a) &\leq mine^{-as}\phi_x(s) \\
s &> 0 \\
&= mine^{-as}e^{-\lambda}e^{\lambda e^s} \\
s &> 0 \\
&= mine^{-\lambda}e^{\lambda e^s - as} \\
s &> 0
\end{aligned}
$$

To find

$$
\begin{aligned}
min(s) &> 0, \\
\frac{d}{ds}e^{\lambda e^s - as} &= e^{\lambda e^s}e^{-as} \times -a + e^{-as}e^{\lambda es} \times \lambda e^s \\
&= e^{\lambda es - as} \times (\lambda e^s - a) \\
&= 0 \\
\Rightarrow a &= \lambda e^s \\
\Rightarrow s &= ln(\frac{a}{\lambda}) \\
Therefore, & \\
P(X > a) &\leq e^{-\lambda}e^{-a(ln\frac{a}{\lambda})}e^{\frac{\lambda \times a}{\lambda}} \\
P(X > a) &\leq (\frac{\lambda}{a})^a e^{a-\lambda}
\end{aligned}
$$

8. **X** is the 3-dimensional Gaussian random vector with expected value $\mu_{\mathbf{X}} = (4, 8, 6)^T$ and co-variance

$$
\begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}
$$

Calculate

(a) The PDF of the first two components of**X**, $f_{X_1X_2}(x_1, x_2)$

(b) The probability thatX_1>8

solution

(a) Let,

$Y = [X_1 \quad X_2]$

$\mu_Y = [4 \quad 8]$

$$C_y = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

$|C_y| = 16 - 4 = 12$

$$C_y^{-1} = \frac{1}{12}. \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$C_y^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$(Y - \mu_Y)^T.C_Y^{-1}(Y - \mu_Y) = \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} y_1 - 4 \\ y_2 - 8 \end{bmatrix}$$

$$= \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16y_1}{3} - \frac{20y_2}{3} + y_2^2 + 112$$

$$f_y(y) = \frac{1}{2\pi\sqrt{12}}.e^{-\frac{1}{2}(y-\mu_y)^T.C_y^{-1}(y-\mu_y)}$$

$$= \frac{1}{\sqrt{48\pi^2}} e^{\frac{-1}{2}.(\frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16y_1}{3} - \frac{20y_2}{3} + y_2^2 + 112)}$$

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{\sqrt{48\pi^2}} e^{\frac{-1}{2}.(\frac{x_1^2}{3} + \frac{x_1 x_2}{3} - \frac{16x_1}{3} - \frac{20x_2}{3} + x_2^2 + 112)}$$

(b) $X_1 - N(4, 4)$

$$\begin{aligned} P(X_1) > 8 &= 1 - P(X_1 \leq 8) \\ &= 1 - G(\frac{8-4}{2}) \\ &= 1 - G(2) \\ &= .0228 \end{aligned}$$